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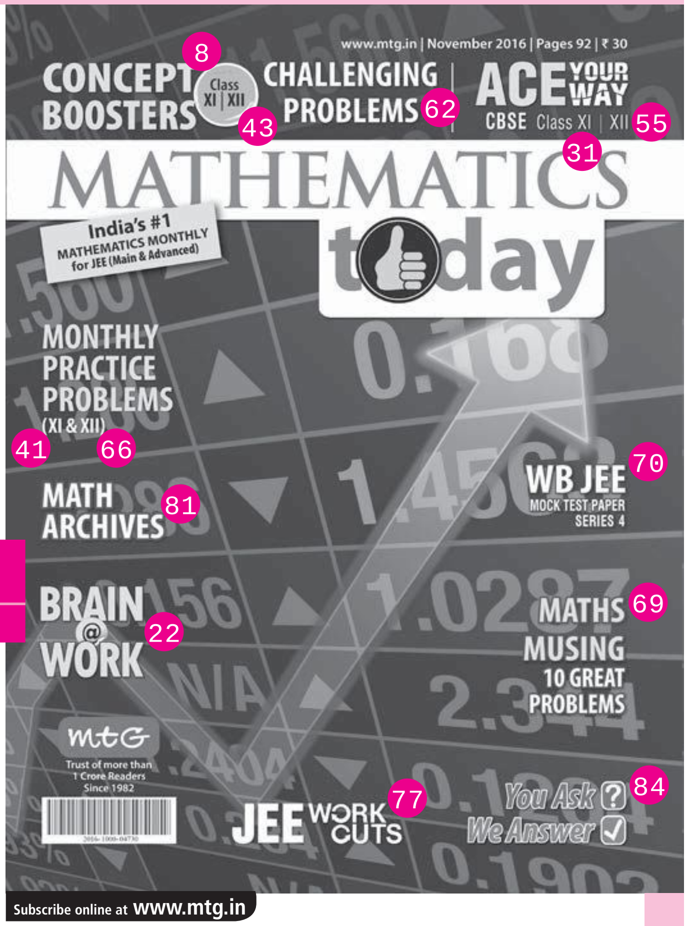

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CONCEPT BOOSTERS

Class XI

Trigonometric Functions

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR, B.Tech, IIT Kanpur

MEASURE OF ANGLES

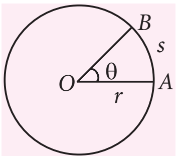
There are three system for measuring angles

1. **Sexagesimal or English system :**
   1 right angle = 90 degree(= 90°)
   1° = 60 minutes (= 60′)
   1′ = 60 seconds (= 60″)
2. **Centesimal or French system :**
   1 right angle = 100 grades $(= 100^g)$
   1 grade = 100 minutes (= 100′)
   1 minute = 100 seconds (= 100″)
3. **Circular system :** The measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

- 1 radian = 57°17′44.8″ ≈ 57°17′45″.
- If s is the length of an arc of a circle of radius r, then the angle θ (in radians) subtended by this arc at the centre of the circle is given by

$$\theta = \frac{s}{r} \text{ or } s = r\theta$$

i.e., Length of arc
= radius × angle in radians



RELATIONSHIP BETWEEN DEGREE, GRADE AND RADIAN

$180° = 200^g = \pi$ radian = 2 Right angles.

i.e. $1° = \left(\frac{10}{9}\right)^g = \left(\frac{\pi}{180}\right)^c$ and $1^c = 57°\,17'44.8''$
$= 63^g 66'20''$

DOMAIN AND RANGE OF A TRIGONOMETRICAL FUNCTION

Trigonometrical Function	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	R
$\text{cosec}\, x$	$R - \{n\pi,\ n \in I\}$	$R - \{x : -1 < x < 1\}$
$\sec x$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$R - \{x : -1 < x < 1\}$
$\cot x$	$R - \{n\pi,\ n \in I\}$	R

TRIGONOMETRICAL RATIOS OF ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°.

Allied angles / Trigo. Ratio	sinθ	cosθ	tanθ
$(-\theta)$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$
$\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta$	$\sin\theta$	$\cot\theta$

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He trains IIT and Olympiad aspirants.

$\left(\frac{\pi}{2}+\theta\right)$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$
$(\pi - \theta)$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$
$(\pi + \theta)$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$
$\left(\frac{3\pi}{2}-\theta\right)$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$
$\left(\frac{3\pi}{2}+\theta\right)$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$
$(2\pi - \theta)$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$
$(2\pi + \theta)$	$\sin\theta$	$\cos\theta$	$\tan\theta$

TRIGONOMETRICAL RATIOS FOR VARIOUS ANGLES

θ	$0°$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	-1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

SUM AND DIFFERENCES OF TWO ANGLES

- $\sin(A + B) = \sin A\cos B + \cos A\sin B$
- $\sin(A - B) = \sin A\cos B - \cos A\sin B$
- $\cos(A + B) = \cos A\cos B - \sin A\sin B$
- $\cos(A - B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$
- $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$
- $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$
- $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$
- $\sin(A + B)\sin(A - B) = \sin^2A - \sin^2B = \cos^2B - \cos^2A$
- $\cos(A + B)\cos(A - B) = \cos^2A - \sin^2B = \cos^2B - \sin^2A$

SUM AND DIFFERENCES OF THREE ANGLES

- $\sin(A + B + C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$
  $= \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\cdot\tan B\cdot\tan C)$
- $\cos(A + B + C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$
  $= \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$
- $\tan(A + B + C)$
  $=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$
- $\cot(A + B + C)$
  $=\dfrac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C\cot A-1}$

In general :

- $\sin(A_1 + A_2 + ... + A_n)$
  $= \cos A_1\cos A_2 ... \cos A_n(S_1 - S_3 + S_5 - S_7 + ...)$
- $\cos(A_1 + A_2 + ... + A_n) = \cos A_1\cos A_2 ... \cos A_n(1 - S_2 + S_4 - S_6 + ...)$
- $\tan(A_1+A_2+....+A_n)=\dfrac{S_1-S_3+S_5-S_7+...}{1-S_2+S_4-S_6+...}$

where, $S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = The sum of the tangents of the separate angles.
$S_2 = \tan A_1\tan A_2 + \tan A_1\tan A_3 + ... + \tan A_{n-1}\tan A_n$ = The sum of the tangent angles taken two at a time.
$S_3 = \tan A_1\tan A_2\tan A_3 + \tan A_2\tan A_3\tan A_4 + ...$ = Sum of tangent angles taken three at a time, and so on.
If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n\tan A$,
$S_2 = {}^nC_2\tan^2A$, $S_3 = {}^nC_3\tan^3A$,...

- $\sin nA = \cos^nA({}^nC_1\tan A - {}^nC_3\tan^3A + {}^nC_5\tan^5A - ...)$
- $\cos nA = \cos^nA(1 - {}^nC_2\tan^2A + {}^nC_4\tan^4A - ...)$
- $\tan nA=\dfrac{{}^nC_1\tan A-{}^nC_3\tan^3A+{}^nC_5\tan^5A-....}{1-{}^nC_2\tan^2A+{}^nC_4\tan^4A-{}^nC_6\tan^6A+...}$
- $\sin(\alpha) + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... +$

$\sin(\alpha + (n - 1)\beta) = \dfrac{\sin\{\alpha+(n-1)(\beta/2)\}\cdot\sin(n\beta/2)}{\sin(\beta/2)}$

- $\cos(\alpha) + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ... +$

$\cos(\alpha + (n - 1)\beta) = \dfrac{\cos\left\{\alpha+(n-1)\left(\frac{\beta}{2}\right)\right\}\cdot\sin\left\{n\left(\frac{\beta}{2}\right)\right\}}{\sin\left(\frac{\beta}{2}\right)}$

PRODUCT INTO SUM OR DIFFERENCE

- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A\sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

SUM AND DIFFERENCE INTO PRODUCTS

- $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$
- $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

- $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
- $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$.

  $=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$
- $\tan A\pm\tan B=\frac{\sin(A\pm B)}{\cos A\cdot\cos B},\left(A\neq n\pi+\frac{\pi}{2},B\neq m\pi\right)$
- $\cot A\pm\cot B=\frac{\sin(B\pm A)}{\sin A\sin B},\left(A\neq n\pi,B\neq m\pi+\frac{\pi}{2}\right)$

TRIGONOMETRIC RATIO OF MULTIPLE OF AN ANGLE

- $\sin 2A=2\sin A\cos A=\frac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

  $=\cos^2 A-\sin^2 A=\frac{1-\tan^2 A}{1+\tan^2 A}$; where $A\neq(2n+1)\frac{\pi}{4}$
- $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$
- $\sin 4A = 4\sin A\cdot\cos^3 A - 4\cos A\sin^3 A$
- $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$
- $\tan 4A=\frac{4\tan A-4\tan^3 A}{1-6\tan^2 A+\tan^4 A}$
- $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$
- $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

MAXIMUM AND MINIMUM VALUE OF $a\sin\theta + b\cos\theta$

Let $a = r\cos\alpha$ ...(i) and $b = r\sin\alpha$ ...(ii)
Squaring and adding (i) and (ii), then $a^2 + b^2 = r^2$ or,

$r=\sqrt{a^2+b^2}$

$\therefore$ $a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$
$= r\sin(\theta + \alpha)$
Since, $-1 \le \sin(\theta + \alpha) \le 1$
Then $-r \le r\sin(\theta + \alpha) \le r$

Hence, $-\sqrt{a^2+b^2}\le a\sin\theta+b\cos\theta\le\sqrt{a^2+b^2}$

Then the greatest and least values of $a\sin\theta + b\cos\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ respectively.

CONDITIONAL TRIGONOMETRICAL IDENTITIES

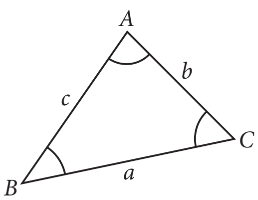
1. If $A + B + C = \pi$, then
   - $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
   - $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$
   - $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A\sin B\sin C$
   - $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$
   - $\cos 2A + \cos 2B - \cos 2C = -1 - 4\sin A\sin B\cos C$
   - $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
   - $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
   - $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
   - $\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   - $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2$
   - $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$
   - $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$
   - $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\cos C$
   - $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
   - $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
   - $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   - $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   - $\tan A + \tan B + \tan C = \tan A\tan B\tan C$
   - $\cot B\cot C + \cot C\cot A + \cot A\cot B = 1$
   - $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$
   - $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
2. If $A+B+C=\frac{\pi}{2}$, then
   - $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\sin C$
   - $\cos^2 A + \cos^2 B + \cos^2 C = 2 + 2\sin A\sin B\sin C$
   - $\sin 2A + \sin 2B + \sin 2C = 4\cos A\cos B\cos C$

VALUES OF TRIGONOMETRICAL RATIOS OF SOME IMPORTANT ANGLES

- $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$
- $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$
- $\tan 15° = 2-\sqrt{3} = \cot 75°$
- $\cot 15° = 2 + \sqrt{3} = \tan 75°$
- $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$
- $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$
- $\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$
- $\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54°$

NAPOLIAN OR NAPIER'S ANALOGY OR TANGENTS LAW

- $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$
- $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$
- $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$



PROBLEMS

Single Correct Answer Type

1. Which of the following relation is correct ?
(a) sin1 < sin1° (b) sin1 > sin1°
(c) sin1 = sin1° (d) $\frac{\pi}{180}\sin 1 = \sin 1°$

2. If $\sin\theta + \text{cosec}\theta = 2$, the value of $\sin^{10}\theta + \text{cosec}^{10}$ is
(a) 10 (b) 2^{10} (c) 2^9 (d) 2

3. If $\sin\theta = \frac{-4}{5}$ and θ lies in the third quadrant, then $\cos\frac{\theta}{2} =$
(a) $\frac{1}{\sqrt{5}}$ (b) $-\frac{1}{\sqrt{5}}$ (c) $\sqrt{\frac{2}{5}}$ (d) $-\sqrt{\frac{2}{5}}$

4. $(m + 2)\sin\theta + (2m - 1)\cos\theta = 2m + 1$, if
(a) $\tan\theta = \frac{3}{4}$ (b) $\tan\theta = \frac{4}{3}$
(c) $\tan\theta = \frac{2m}{m^2+1}$ (d) None of these

5. If $\sin x + \sin y = 3(\cos y - \cos x)$, then the value of $\frac{\sin 3x}{\sin 3y}$ is
(a) 1 (b) –1
(c) 0 (d) None of these

6. If $\tan\theta + \sec\theta = e^x$, then $\cos\theta$ equals
(a) $\frac{(e^x+e^{-x})}{2}$ (b) $\frac{2}{(e^x+e^{-x})}$
(c) $\frac{(e^x-e^{-x})}{2}$ (d) $\frac{(e^x-e^{-x})}{(e^x+e^{-x})}$

7. If $x+\frac{1}{x} = 2\cos\alpha$, then $x^n + \frac{1}{x^n} =$
(a) $2^n\cos\alpha$ (b) $2^n\cos n\alpha$
(c) $2i\sin n\alpha$ (d) $2\cos n\alpha$

8. $\frac{1+\sin A-\cos A}{1+\sin A+\cos A} =$
(a) $\sin\frac{A}{2}$ (b) $\cos\frac{A}{2}$ (c) $\tan\frac{A}{2}$ (d) $\cot\frac{A}{2}$

9. If $2y\cos\theta = x\sin\theta$ and $2x\sec\theta - y\,\text{cosec}\theta = 3$, then $x^2 + 4y^2 =$
(a) 4 (b) –4
(c) 0 (d) None of these

10. If $x = \sec\phi - \tan\phi$, $y = \text{cosec}\phi + \cot\phi$, then
(a) $x = \frac{y+1}{y-1}$ (b) $x = \frac{y-1}{y+1}$
(c) $y = \frac{1-x}{1+x}$ (d) None of these

11. If $\tan\theta = \frac{x\sin\phi}{1-x\cos\phi}$ and $\tan\phi = \frac{y\sin\theta}{1-y\cos\theta}$, then $\frac{x}{y} =$
(a) $\frac{\sin\phi}{\sin\theta}$ (b) $\frac{\sin\theta}{\sin\phi}$
(c) $\frac{\sin\phi}{1-\cos\theta}$ (d) $\frac{\sin\theta}{1-\cos\phi}$

12. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then
(a) $m^2 - n^2 = 4mn$ (b) $m^2 + n^2 = 4mn$
(c) $m^2 - n^2 = m^2 + n^2$ (d) $m^2 - n^2 = 4\sqrt{mn}$

13. If $\cot\theta + \tan\theta = m$ and $\sec\theta - \cos\theta = n$, then which of the following is correct
(a) $m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1$
(b) $m(m^2n)^{1/3} - n(mn^2)^{1/3} = 1$
(c) $n(mn^2)^{1/3} - m(nm^2)^{1/3} = 1$
(d) $n(m^2n)^{1/3} - m(mn^2)^{1/3} = 1$

14. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12}x + 3\cos^{10}x + 3\cos^8 x + \cos^6 x - 2$ is equal to
(a) 0 (b) 1 (c) – 1 (d) 2

15. If $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$ and $x\sin\alpha - y\cos\alpha = 0$, then $x^2 + y^2 =$
(a) – 1 (b) ±1
(c) 1 (d) None of these

16. If $\sin^2\theta = \dfrac{x^2+y^2+1}{2x}$, then x must be
(a) – 3 (b) – 2
(c) 1 (d) None of these

17. If $\tan\theta - \cot\theta = a$ and $\sin\theta + \cos\theta = b$, then $(b^2 - 1)^2(a^2 + 4)$ is equal to
(a) 2 (b) – 4 (c) 3 (d) 4

18. If $\tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha + 2\tan^2\alpha\tan^2\beta\tan^2\gamma = 1$ then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is
(a) 0 (b) – 1
(c) 1 (d) None of these

19. $\cos1° + \cos2° + \cos3° + \ldots + \cos180°$ equals
(a) 1 (b) –1
(c) 0 (d) None of the above

20. If angle θ be divided into two parts such that the tangent of one part is k times the tangent of the other and ϕ is their difference, then $\sin\theta =$
(a) $\dfrac{k+1}{k-1}\sin\phi$ (b) $\dfrac{k-1}{k+1}\sin\phi$
(c) $\dfrac{2k-1}{2k+1}\sin\phi$ (d) None of these

21. Given that $\pi < \alpha < \dfrac{3\pi}{2}$, then the expression $\sqrt{(4\sin^4\alpha + \sin^2 2\alpha)} + 4\cos^2\left(\dfrac{\pi}{4} - \dfrac{\alpha}{2}\right)$ is equal to
(a) 2 (b) $2 + 4\sin\alpha$
(c) $2 - 4\sin\alpha$ (d) None of these

22. $\tan100° + \tan125° + \tan100°\tan125° =$
(a) 0 (b) $\dfrac{1}{2}$ (c) – 1 (d) 1

23. If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$, then each side is equal to
(a) $\pm\sin A\sin B\sin C$ (b) $\pm\cos A\cos B\cos C$
(c) $\pm\sin A\cos B\cos C$ (d) $\pm\cos A\sin B\sin C$

24. If $\tan\alpha = (1 + 2^{-x})^{-1}$, $\tan\beta = (1 + 2^{x+1})^{-1}$, then $\alpha + \beta$ equals
(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

25. The sum $S = \sin\theta + \sin2\theta + \ldots + \sin n\theta$, equals
(a) $\sin\frac{1}{2}(n+1)\,\theta\cdot\sin\frac{1}{2}n\theta/\sin\frac{\theta}{2}$
(b) $\cos\frac{1}{2}(n+1)\,\theta\cdot\sin\frac{1}{2}n\theta/\sin\frac{\theta}{2}$
(c) $\sin\frac{1}{2}(n+1)\theta\cdot\cos\frac{1}{2}n\theta/\sin\frac{\theta}{2}$
(d) $\cos\frac{1}{2}(n+1)\theta\cdot\cos\frac{1}{2}n\theta/\sin\frac{\theta}{2}$

26. The value of $\cot70° + 4\cos70°$ is
(a) $\dfrac{1}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\dfrac{1}{2}$

27. $\tan20°\tan40°\tan60°\tan80° =$
(a) 1 (b) 2 (c) 3 (d) $\sqrt{3}/2$

28. $\sin36°\sin72°\sin108°\sin144° =$
(a) 1/4 (b) 1/16 (c) 3/4 (d) 5/16

29. If $x = \cos10°\cos20°\cos40°$, then the value of x is
(a) $\dfrac{1}{4}\tan10°$ (b) $\dfrac{1}{8}\cot10°$
(c) $\dfrac{1}{8}\mathrm{cosec}10°$ (d) $\dfrac{1}{8}\sec10°$

30. $\tan9° - \tan27° - \tan63° + \tan81° =$
(a) 1/2 (b) 2 (c) 4 (d) 8

31. $\dfrac{\sin3\theta + \sin5\theta + \sin7\theta + \sin9\theta}{\cos3\theta + \cos5\theta + \cos7\theta + \cos9\theta} =$
(a) $\tan3\theta$ (b) $\cot3\theta$ (c) $\tan6\theta$ (d) $\cot6\theta$

32. $\dfrac{\sin(B+A) + \cos(B-A)}{\sin(B-A) + \cos(B+A)} =$
(a) $\dfrac{\cos B + \sin B}{\cos B - \sin B}$ (b) $\dfrac{\cos A + \sin A}{\cos A - \sin A}$
(c) $\dfrac{\cos A - \sin A}{\cos A + \sin A}$ (d) None of these

33. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$ then $\dfrac{m+n}{m-n} =$
(a) $2\cos2\theta$ (b) $\cos2\theta$ (c) $2\sin2\theta$ (d) $\sin2\theta$

34. The value of $\tan20° + 2\tan50° - \tan70°$ is equal to
(a) 1 (b) 0
(c) $\tan50°$ (d) None of these

35. If $\cos2B = \dfrac{\cos(A+C)}{\cos(A-C)}$, then $\tan A$, $\tan B$, $\tan C$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

Multiple Correct Answer Type

36. If $\left(\frac{\sin\theta}{\sin\phi}\right)^2 = \frac{\tan\theta}{\tan\phi} = 3$ then

(a) $\tan\phi = \frac{1}{\sqrt{3}}$ (b) $\tan\phi = -\frac{1}{\sqrt{3}}$

(c) $\tan\theta = \sqrt{3}$ (d) $\tan\theta = -\sqrt{3}$

37. For $\alpha = \frac{\pi}{7}$ which of the following hold(s) good?

(a) $\tan\alpha\tan2\alpha\tan3\alpha = \tan3\alpha - \tan2\alpha - \tan\alpha$
(b) $\operatorname{cosec}\alpha = \operatorname{cosec}2\alpha + \operatorname{cosec}4\alpha$
(c) $\cos\alpha - \cos2\alpha + \cos3\alpha = \frac{1}{2}$
(d) $8\cos\alpha\cos2\alpha\cos4\alpha = 1$

38. Which of the following quantities are rational?

(a) $\sin\left(\frac{11\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right)$

(b) $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right)$

(c) $\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)$

(d) $\left(1+\cos\frac{2\pi}{9}\right)\left(1+\cos\frac{4\pi}{9}\right)\left(1+\cos\frac{8\pi}{9}\right)$

39. For $0 \le x \le 2\pi$ then $2^{\operatorname{cosec}^2 x}\sqrt{\frac{y^2}{2} - y + 1} \le \sqrt{2}$ is

(a) satisfied by exactly one value of y
(b) satisfied by exactly two values of x
(c) satisfied by x for which $\cos x = 0$
(d) satisfied by x for which $\sin x = 0$

40. If $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta = 2a$ and $2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) = 1$ then

(a) $\cos\alpha + \cos\beta = \cos\alpha\cos\beta$

(b) $\cos\alpha\cdot\cos\beta = \frac{4a^2+y^2}{x^2+y^2}$

(c) $\cos\alpha + \cos\beta = \frac{4ax}{x^2+y^2}$

(d) none of these

41. Which of the following is true for a ΔABC

(a) $R^2 \ge \left(\frac{abc}{a+b+c}\right)$

(b) $r + 2R = s$ if $C = 90°$

(c) $\sin2A + \sin2B + \sin2C \le \frac{3\sqrt{3}}{2}$

(d) $\frac{2}{R} \le \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

Comprehension Type

Paragraph for Q. No. 42 to 44

Let $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ are the roots of equation $8x^3 - 4x^2 - 4x + 1 = 0$

42. The value of $\sec\left(\frac{\pi}{7}\right) + \sec\left(\frac{3\pi}{7}\right) + \sec\left(\frac{5\pi}{7}\right)$ is

(a) 2 (b) 4
(c) 8 (d) None of these

43. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

44. The value of $\cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right)$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

Paragraph for Q. No. 45 to 47

$\sin\alpha + \sin\beta = \frac{1}{4}$ and $\cos\alpha + \cos\beta = \frac{1}{3}$

45. The value of $\sin(\alpha + \beta)$ is

(a) $\frac{24}{25}$ (b) $\frac{13}{25}$ (c) $\frac{12}{13}$ (d) None of these

46. The value of $\cos(\alpha + \beta)$ is

(a) $\frac{12}{25}$ (b) $\frac{7}{25}$ (c) $\frac{12}{13}$ (d) None of these

47. The value of $\tan(\alpha + \beta)$ is

(a) $\frac{25}{7}$ (b) $\frac{25}{12}$ (c) $\frac{25}{13}$ (d) $\frac{24}{7}$

Matrix-Match Type

48. Match the following trigonometric ratios with the equations whose one of the roots is given

Column I		Column II	
A.	cos20°	P.	$x^3 - 3x^2 - 3x + 1 = 0$
B.	sin10°	Q.	$32x^5 - 40x^3 + 10x - 1 = 0$
C.	tan15°	R.	$8x^3 - 6x - 1 = 0$
D.	sin6°	S.	$8x^3 - 6x + 1 = 0$

49. Match the following :

	Column I		Column II
A.	The maximum value of $\cos(2A+\theta)+\cos(2B+\theta)$ ($\theta \in R$ and A, B are constants)	P.	$2\sin(A+B)$
B.	Maximum value of $\cos 2A + \cos 2B$ $\left(A, B \in \left(0, \frac{\pi}{2}\right), A+B\right.$ is constant)	Q.	$2\sec(A+B)$
C.	Minimum value of $\sec 2A + \sec 2B$ $\left(A, B \in \left(0, \frac{\pi}{4}\right), A+B\right.$ is constant)	R.	$2\cos(A+B)$
D.	Minimum value of $\sqrt{\tan\theta + \cot\theta - 2\cos(2A+2B)}$ ($\theta \in R$, A, B are constants)	S.	$2\cos(A-B)$

50. Match the following :

	Column I		Column II
A.	If $\Delta = a^2 - (b-c)^2$, where Δ is the area of the triangle ABC, then $\tan A$ is equal to	P.	$\frac{1}{2}$
B.	In a ΔABC, given that $\tan A : \tan B : \tan C = 3:4:5$, then the value of $\sin A \sin B \sin C$ is	Q.	$\frac{8}{15}$
C.	Let $f(\theta) = \frac{1-\sin 2\theta + \cos 2\theta}{2\cos 2\theta}$ and $\alpha+\beta = \frac{5\pi}{4}$, then the value of $f(\alpha)\, f(\beta)$ is	R.	$\frac{\pi}{4}$
		S.	$\frac{2\sqrt{5}}{7}$

Integer Answer Type

51. If $\frac{\tan x}{2} = \frac{\tan y}{3} = \frac{\tan z}{5}$ and $x + y + z = \pi$, $\tan^2 x + \tan^2 y + \tan^2 z = \frac{38}{K}$ then $K =$

52. If $\tan\alpha$ is an integral solution of $4x^2 - 16x + 15 < 0$ and $\cos\beta$ is the slope of the bisector of the angle in the first quadrant between the x and y axis. Then $\sin(\alpha+\beta) : \sin(\alpha-\beta) =$

53. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then the value of $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta$ must be

54. In a ΔABC, $a = 5$, $b = 4$ and $\cos(A-B) = \frac{31}{32}$, then c must be

55. In a triangle ABC, if r_1, r_2, r_3 are the ex-radius then $\frac{bc}{r_1} + \frac{ac}{r_2} + \frac{ab}{r_3} = k\frac{abc}{2\Delta}\left[\frac{s}{a} + \frac{s}{b} + \frac{s}{c} - 3\right]$ then k is equal to

SOLUTIONS

1. (b) : The true relation is $\sin 1 > \sin 1°$
Since, value of $\sin\theta$ is increasing

2. (d) : We have, $\sin\theta + \operatorname{cosec}\theta = 2$

$\Rightarrow \sin^2\theta + 1 = 2\sin\theta \Rightarrow \sin^2\theta - 2\sin\theta + 1 = 0$

$\Rightarrow (\sin\theta - 1)^2 = 0 \Rightarrow \sin\theta = 1$

$\Rightarrow \sin^{10}\theta + \operatorname{cosec}^{10}\theta = (1)^{10} + \frac{1}{(1)^{10}} = 2$

3. (b) : Given that $\sin\theta = -\frac{4}{5}$ and θ lies in the III quadrant.

$\Rightarrow \cos\theta = \sqrt{1 - \frac{16}{25}} = -\frac{3}{5}$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1-3/5}{2}} = \pm\sqrt{\frac{1}{5}}$$

Since $\theta/2$ will be in III quadrant.

Hence, $\cos\frac{\theta}{2} = -\frac{1}{\sqrt{5}}$.

4. (b) : Squaring the given relation and putting $\tan\theta = t$

$$(m+2)^2 t^2 + 2(m+2)(2m-1)t + (2m-1)^2 = (2m+1)^2(1+t^2)$$

$\Rightarrow 3(1-m^2)t^2 + (4m^2+6m-4)t - 8m = 0$

$\Rightarrow (3t-4)[(1-m^2)t + 2m] = 0$,

which is true if $t = \tan\theta = \frac{4}{3}$ or $\tan\theta = \frac{2m}{m^2-1}$

5. (b) : We have $\sin x + \sin y = 3(\cos y - \cos x)$

$\Rightarrow \sin x + 3\cos x = 3\cos y - \sin y$ ...(i)

$\Rightarrow r\cos(x-\alpha) = r\cos(y+\alpha)$,

where $r = \sqrt{10}$, $\tan\alpha = \frac{1}{3}$

$\Rightarrow x - \alpha = \pm(y+\alpha) \Rightarrow x = -y$ or $x - y = 2\alpha$

Clearly, $x = -y$ satisfies (i)

$\therefore \frac{\sin 3x}{\sin 3y} = \frac{-\sin 3y}{\sin 3y} = -1$

6. (b) : We have, $\tan\theta + \sec\theta = e^x$ ...(i)

$\therefore \sec\theta - \tan\theta = e^{-x}$ ...(ii)

From (i) and (ii), we get

$$2\sec\theta = e^x + e^{-x} \Rightarrow \cos\theta = \frac{2}{e^x + e^{-x}}.$$

7. (d) : We have, $x+\frac{1}{x}=2\cos\alpha$

$\Rightarrow\quad x^2+\frac{1}{x^2}=4\cos^2\alpha-2$

$\Rightarrow\quad x^2+\frac{1}{x^2}=2(2\cos^2\alpha-1)=2\cos 2\alpha$

Similarly, $x^n+\frac{1}{x^n}=2\cos n\alpha$

8. (c) : $\frac{1+\sin A-\cos A}{1+\sin A+\cos A}=\frac{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}$

$$=\frac{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}{2\cos\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)}=\tan\frac{A}{2}$$

9. (a) : Given that $2y\cos\theta = x\sin\theta$ ...(i)

and $2x\sec\theta - y\,\text{cosec}\,\theta = 3$ ...(ii)

$\Rightarrow\quad \frac{2x}{\cos\theta}-\frac{y}{\sin\theta}=3$

$\Rightarrow\quad 2x\sin\theta - y\cos\theta - 3\sin\theta\cos\theta = 0$ ...(iii)

Solving (i) and (iii), we get $y = \sin\theta$ and $x = 2\cos\theta$

Now, $x^2 + 4y^2 = 4\cos^2\theta + 4\sin^2\theta = 4$

10. (b) : We have, $xy = (\sec\phi - \tan\phi)(\text{cosec}\,\phi + \cot\phi)$

$$=\frac{1-\sin\phi}{\cos\phi}\cdot\frac{1+\cos\phi}{\sin\phi}$$

$\Rightarrow\quad xy+1=\frac{1-\sin\phi+\cos\phi-\sin\phi\cos\phi+\sin\phi\cos\phi}{\cos\phi\sin\phi}$

$=\frac{1-\sin\phi+\cos\phi}{\cos\phi\sin\phi}$ ...(i)

Also, $x - y = (\sec\phi - \tan\phi) - (\text{cosec}\,\phi + \cot\phi)$

$=\frac{1-\sin\phi}{\cos\phi}-\frac{1+\cos\phi}{\sin\phi}=\frac{\sin\phi-\sin^2\phi-\cos\phi-\cos^2\phi}{\cos\phi\sin\phi}$

$=\frac{\sin\phi-\cos\phi-1}{\cos\phi\sin\phi}$ ...(ii)

Adding (i) and (ii) we get, $xy + 1 + (x - y) = 0$

$\Rightarrow\quad x=\frac{y-1}{y+1}$

11. (b) : We have $\tan\theta=\frac{x\sin\phi}{1-x\cos\phi}$

$\Rightarrow\quad x\sin\phi = \tan\theta - x\cos\phi\tan\theta$

$\Rightarrow\quad x=\frac{\tan\theta}{\sin\phi+\cos\phi\tan\theta}$

$$=\frac{\sin\theta}{\cos\theta\sin\phi+\cos\phi\sin\theta}=\frac{\sin\theta}{\sin(\theta+\phi)}$$

Similarly, $y=\frac{\sin\phi}{\sin(\theta+\phi)}$; $\therefore\ \frac{x}{y}=\frac{\sin\theta}{\sin\phi}$.

12. (d) : $(m + n) = 2\tan\theta$, $(m - n) = 2\sin\theta$

$\therefore\quad m^2 - n^2 = 4\tan\theta\cdot\sin\theta$ ...(i)

and $4\sqrt{mn}=4\sqrt{\tan^2\theta-\sin^2\theta}=4\sin\theta\cdot\tan\theta$ ...(ii)

From (i) and (ii), $m^2-n^2=4\sqrt{mn}$

13. (a) : As given,

$\frac{1}{\tan\theta}+\tan\theta=m\ \Rightarrow\ 1+\tan^2\theta=m\tan\theta$

$\Rightarrow\quad \sec^2\theta = m\tan\theta$ ...(i)

and $\sec\theta - \cos\theta = n\ \Rightarrow\ \sec^2\theta - 1 = n\sec\theta$

$\Rightarrow\quad \tan^2\theta = n\sec\theta$

$\Rightarrow\quad \tan^4\theta = n^2\sec^2\theta = n^2 m\tan\theta$ {by (i)}

$\Rightarrow\quad \tan^3\theta = n^2m$ [$\because \tan\theta \neq 0$]

$\Rightarrow\quad \tan\theta = (n^2m)^{1/3}$ ...(ii)

Also, $\sec^2\theta = m\tan\theta = m(n^2m)^{1/3}$ {by (i) and (ii)}

$\Rightarrow\quad m(mn^2)^{1/3} - (n^2m)^{2/3} = 1$

$\Rightarrow\quad m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1$

14. (c) : We have, $\sin x + \sin^2 x = 1$

or $\sin x = 1 - \sin^2 x$ or $\sin x = \cos^2 x$

$\therefore\quad \cos^{12}x + 3\cos^{10}x + 3\cos^8x + \cos^6x - 2$

$= \sin^6x + 3\sin^5x + 3\sin^4x + \sin^3x - 2$

$= (\sin^2x)^3 + 3(\sin^2x)^2\sin x + 3(\sin^2x)(\sin x)^2 + (\sin x)^3 - 2$

$= (\sin^2x + \sin x)^3 - 2 = (1)^3 - 2 = -1.$

15. (c) : We have, $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$ ... (i)

and $x\sin\alpha - y\cos\alpha = 0$ ... (ii)

From (i) and (ii), we get

$\Rightarrow\quad y\cos\alpha\sin^2\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$

$\Rightarrow\quad y\cos\alpha\{\sin^2\alpha + \cos^2\alpha\} = \sin\alpha\cos\alpha$

$\Rightarrow\quad y\cos\alpha = \sin\alpha\cos\alpha\ \Rightarrow\ y = \sin\alpha$ and $x = \cos\alpha$

Hence, $x^2 + y^2 = \sin^2\alpha + \cos^2\alpha = 1$

16. (d) : Since, $\sin^2\theta \leq 1$

$\therefore\quad \frac{x^2+y^2+1}{2x}\leq 1$

$x^2 + y^2 - 2x + 1 \leq 0$

$(x - 1)^2 + y^2 \leq 0$

It is possible, iff $x = 1$ and $y = 0$,

i.e., It also depends on value of y.

17. (d) : Given that $\tan\theta - \cot\theta = a$ ...(i)

and $\sin\theta + \cos\theta = b$ ...(ii)

Now, $(b^2 - 1)^2 (a^2 + 4)$

$= \{(\sin\theta + \cos\theta)^2 - 1\}^2 \{(\tan\theta - \cot\theta)^2 + 4\}$

$= [1 + \sin 2\theta - 1]^2 [\tan^2\theta + \cot^2\theta - 2 + 4]$

$= \sin^2 2\theta(\text{cosec}^2\theta + \sec^2\theta)$

$$=4\sin^2\theta\cos^2\theta\left[\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right]=4$$

18. (c) : $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$$=\frac{\tan^2\alpha}{1+\tan^2\alpha}+\frac{\tan^2\beta}{1+\tan^2\beta}+\frac{\tan^2\gamma}{1+\tan^2\gamma}$$

$$=\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}$$

(Here, $x=\tan^2\alpha, y=\tan^2\beta, z=\tan^2\gamma$)

$$=\frac{(x+y+z)+(xy+yz+zx+2xyz)+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$$

$$=\frac{1+x+y+z+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}=1$$

($\because$ $xy + yz + zx + 2xyz = 1$)

19. (b) : $(\cos1° + \cos179°) + (\cos2° + \cos178°) + ... + (\cos89° + \cos91°) + \cos90° + \cos180° = -1$

20. (a) : Let $A + B = \theta$ and $A - B = \phi$.

Then $\tan A = k\tan B$ or $\frac{k}{1}=\frac{\tan A}{\tan B}=\frac{\sin A\cos B}{\cos A\sin B}$

Applying componendo and dividendo

$$\Rightarrow \frac{k+1}{k-1}=\frac{\sin A\cos B+\cos A\sin B}{\sin A\cos B-\cos A\sin B}=\frac{\sin(A+B)}{\sin(A-B)}=\frac{\sin\theta}{\sin\phi}$$

$$\Rightarrow \sin\theta=\frac{k+1}{k-1}\sin\phi$$

21. (c) : Given that $\pi<\alpha<\frac{3\pi}{2}$ *i.e.*, α is in third quadrant.

Now, $\sqrt{(4\sin^4\alpha+\sin^2 2\alpha)}+4\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

$$=\sqrt{(4\sin^4\alpha+4\sin^2\alpha\cos^2\alpha)}+2\cdot2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$=\sqrt{4\sin^2\alpha(\sin^2\alpha+\cos^2\alpha)}+2\left[1+\cos\left(\frac{\pi}{2}-\alpha\right)\right]$$

$= \pm\, 2\sin\alpha + 2 + 2\sin\alpha$

On taking –ve, answer is 2 and on taking +ve, answer is $2 + 4\sin\alpha$.

But $\pi<\alpha<\frac{3\pi}{2}$, Hence answer is $2 - 4\sin\alpha$ because $\sin\alpha$ is –ve in third quadrant.

22. (d) : $\tan(100°+125°)=\frac{\tan100°+\tan125°}{1-\tan100°\tan125°}$

$$\therefore\ \tan225°=\frac{\tan 100°+\tan 125°}{1-\tan100°\tan125°}$$

i.e., $1=\frac{\tan100°+\tan 125°}{1-\tan 100°\tan 125°}$

i.e., $\tan100° + \tan125° + \tan100° \tan125° = 1$

23. (b) : Multiplying both sides by $(1 - \sin A)(1 - \sin B)(1 - \sin C)$, we have,

$(1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 C)$

$= (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$

$\Rightarrow$ $(1 - \sin A)(1 - \sin B)(1 - \sin C) = \pm\cos A \cos B \cos C$

Similarly,

$(1 + \sin A)(1 + \sin B)(1 + \sin C) = \pm\cos A \cos B \cos C$

24. (b) : $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$\Rightarrow\ \tan(\alpha+\beta)=\frac{\frac{1}{1+\frac{1}{2^x}}+\frac{1}{1+2^{x+1}}}{1-\frac{1}{1+1/2^x}\times\frac{1}{1+2^{x+1}}}$$

$$\Rightarrow\ \tan(\alpha+\beta)=\frac{2^x+2\cdot2^{x+x}+2^x+1}{1+2^x+2\cdot2^x+2\cdot2^{x+x}-2^x}$$

$$\Rightarrow\ \tan(\alpha+\beta)=1\ \Rightarrow\ \alpha+\beta=\frac{\pi}{4}$$

25. (a) : $S = \sin\theta + \sin2\theta + \sin3\theta + ... + \sin n\theta$

We know, $\sin\theta + \sin(\theta + \beta) + \sin(\theta + 2\beta) + ...$ n terms

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\theta+(n-1)\frac{\beta}{2}\right]$$

Put $\beta = \theta$, then $S=\frac{\sin\frac{n\theta}{2}.\sin\frac{\theta(n+1)}{2}}{\sin\frac{\theta}{2}}$

26. (b) : Now,

$$\cot70°+4\cos70°=\frac{\cos70°+4\sin70°\cos70°}{\sin70°}$$

$$=\frac{\cos70°+2\sin140°}{\sin70°}=\frac{\cos70°+2\sin(180°-40°)}{\sin70°}$$

$$=\frac{\sin20°+\sin40°+\sin40°}{\sin70°}=\frac{2\sin30°\cos10°+\sin40°}{\sin70°}$$

$$=\frac{\sin80°+\sin40°}{\sin70°}=\frac{2\sin60°\cos20°}{\sin70°}=\sqrt{3}$$

27. (c) : $\tan20° \tan40° \tan60° \tan80°$

$$=\frac{\sin20°\sin 40°\sin 80°\tan60°}{\cos 20°\cos 40°\cos 80°}$$

Let $N^r = (\sin20° \sin40° \sin80°)$

$$=\frac{\sin 20°}{2}(2\sin 40°\sin 80°)$$

$$=\frac{\sin20°}{2}(\cos40°-\cos120°)$$

$$=\frac{1}{2}\sin 20°\left(1-2\sin^2 20°+\frac{1}{2}\right)$$

$$=\frac{1}{2}\sin 20°\left(\frac{3}{2}-2\sin^2 20°\right)=\frac{\sin 60°}{4}=\frac{\sqrt{3}}{8}$$

Now, we take D^r = cos20° cos40° cos80°

$$=\frac{\sin 2^3 20°}{2^3\sin 20°}=\frac{\sin 160°}{8\sin 20°}=\frac{\sin 20°}{8\sin 20°}=\frac{1}{8}$$

$\therefore$ Hence $\tan 20°\tan 40°\tan 80°=\dfrac{\sqrt{3}/8}{1/8}$

Therefore tan20° tan40° tan60° tan80° = $\sqrt{3}\cdot\sqrt{3}=3$

28. (d) : sin36° sin72° sin108° sin144°

$$=\sin^2 36°\sin^2 72°=\frac{1}{4}\{(2\sin^2 36°)(2\sin^2 72°)\}$$

$$=\frac{1}{4}\{(1-\cos 72°)(1-\cos 144°)\}$$

$$=\frac{1}{4}\{(1-\sin 18°)(1+\cos 36°)\}$$

$$=\frac{1}{4}\left[\left(1-\frac{\sqrt{5}-1}{4}\right)\left(1+\frac{\sqrt{5}+1}{4}\right)\right]=\frac{20}{16}\times\frac{1}{4}=\frac{5}{16}$$

29. (b) : x = cos10° cos20° cos40°

$$=\frac{1}{2\sin 10°}[2\sin 10°\cos 10°\cos 20°\cos 40°]$$

$$=\frac{1}{2\cdot 2\sin 10°}[2\sin 20°\cos 20°\cos 40°]$$

$$=\frac{1}{2\cdot 4\sin 10°}[2\sin 40°\cos 40°]=\frac{1}{8\sin 10°}(\sin 80°)$$

$$=\frac{1}{8\sin 10°}\cos 10°=\frac{1}{8}\cot 10°$$

30. (c) : tan9° – tan27° – tan63° + tan81°

= tan9° – tan27° – cot27° + cot9°

= (tan9° + cot9°) – (tan27° + cot27°)

$$=\frac{2}{\sin 18°}-\frac{2}{\sin 54°}$$

$$=2\left\{\frac{\sin 54°-\sin 18°}{\sin 18°\sin 54°}\right\}=2.\frac{2\cdot\cos 36°\cdot\sin 18°}{\sin 18°\sin 54°}=4$$

31. (c) : $\dfrac{\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta}{\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 9\theta}$

$$=\frac{(\sin 3\theta+\sin 9\theta)+(\sin 5\theta+\sin 7\theta)}{(\cos 3\theta+\cos 9\theta)+(\cos 5\theta+\cos 7\theta)}$$

$$=\frac{2\sin 6\theta\cos 3\theta+2\sin 6\theta\cos\theta}{2\cos 6\theta\cos 3\theta+2\cos 6\theta\cos\theta}$$

$$=\frac{2\sin 6\theta(\cos 3\theta+\cos\theta)}{2\cos 6\theta(\cos 3\theta+\cos\theta)}=\tan 6\theta$$

32. (b) : $\dfrac{\sin(B+A)+\cos(B-A)}{\sin(B-A)+\cos(B+A)}$

$$=\frac{\sin(B+A)+\sin(90°-\overline{B-A})}{\sin(B-A)+\sin(90°-\overline{A+B})}$$

$$=\frac{2\sin(A+45°)\cos(45°-B)}{2\sin(45°-A)\cos(45°-B)}$$

$$=\frac{\sin(A+45°)}{\sin(45°-A)}=\frac{\cos A+\sin A}{\cos A-\sin A}$$

33. (a) : $\dfrac{m}{n}=\dfrac{\tan(120°+\theta)}{\tan(\theta-30°)}$

$$\Rightarrow\ \frac{m+n}{m-n}=\frac{\tan(\theta+120°)+\tan(\theta-30°)}{\tan(\theta+120°)-\tan(\theta-30°)}$$

(By componendo and dividendo)

$$=\frac{\sin(\theta+120°)\cos(\theta-30°)+\cos(\theta+120°)\sin(\theta-30°)}{\sin(\theta+120°)\cos(\theta-30°)-\cos(\theta+120°)\sin(\theta-30°)}$$

$$=\frac{\sin(2\theta+90°)}{\sin(150°)}=\frac{\cos 2\theta}{1/2}=2\cos 2\theta$$

34. (b) : tan20° + 2tan50° – tan70°

$$=\frac{\sin 20°}{\cos 20°}-\frac{\sin 70°}{\cos 70°}+2\tan 50°$$

$$=\frac{\sin 20°\cos 70°-\cos 20°\sin 70°}{\cos 20°\cos 70°}+2\tan 50°$$

$$=\frac{\sin(20°-70°)}{\frac{1}{2}[\cos(70°+20°)+\cos(70°-20°)]}+2\tan 50°$$

$$=\frac{2\sin(-50°)}{\cos 90°+\cos 50°}+2\tan 50°$$

$$=\frac{-2\sin 50°}{0+\cos 50°}+2\tan 50°$$

= –2tan50° + 2tan50° = 0

35. (b) : We have,

$$\cos 2B=\frac{\cos(A+C)}{\cos(A-C)}=\frac{\cos A\cos C-\sin A\sin C}{\cos A\cos C+\sin A\sin C}$$

$$\Rightarrow\ \frac{1-\tan^2 B}{1+\tan^2 B}=\frac{1-\tan A\tan C}{1+\tan A\tan C}$$

$$\Rightarrow\ 1+\tan^2 B-\tan A\tan C-\tan A\tan C\tan^2 B$$
$$=1-\tan^2 B+\tan A\tan C-\tan A\tan C\tan^2 B$$

$$\Rightarrow\ 2\tan^2 B=2\tan A\tan C\ \Rightarrow\ \tan^2 B=\tan A\tan C$$

Hence, tan A, tan B and tan C will be in G.P.

36. (a, b, c, d) : Since, $\left(\frac{\sin\theta}{\sin\phi}\right)^2 = \frac{\tan\theta}{\tan\phi}$

$$\Rightarrow \frac{\sin\theta}{\sin\phi}\frac{\sin\theta}{\sin\phi} = \frac{\sin\theta}{\sin\phi}\frac{\cos\phi}{\cos\theta}$$

$$\Rightarrow \frac{\sin\theta}{\sin\phi} = \frac{\cos\phi}{\cos\theta} \Rightarrow \sin 2\theta = \sin 2\phi$$

$$\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\phi}{1+\tan^2\phi} \Rightarrow \frac{6\tan\phi}{1+9\tan^2\phi} = \frac{2\tan\phi}{1+\tan^2\phi}$$

$$\Rightarrow \tan^2\phi = \frac{1}{3} \text{ and } \tan\theta = \pm\sqrt{3}$$

37. (a, b, c) :

(a) $3\alpha = 2\alpha + \alpha$

$\Rightarrow$ $\tan 3\alpha = \tan(2\alpha + \alpha)$

$\Rightarrow$ $\tan 3\alpha - \tan 2\alpha - \tan\alpha = \tan\alpha \tan 2\alpha \tan 3\alpha$

(b) $$\text{R.H.S} = \frac{\sin 4\alpha + \sin 2\alpha}{\sin 2\alpha \sin 4\alpha} = \frac{2\sin 3\alpha\cos\alpha}{\sin 2\alpha\sin 4\alpha}$$

$$= \frac{2\sin\frac{3\pi}{7}\cos\frac{\pi}{7}}{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}} = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$$

(c) $$\cos\alpha - \cos 2\alpha + \cos 3\alpha = \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}$$

$$= 1/2$$

(d) $8\cos\alpha \cos 2\alpha \cos 4\alpha = -1$

38. (a, b, c, d) :

(a) $$\sin\frac{11\pi}{12}\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{6}\right) = \frac{1}{4} \in Q$$

(b) $$\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right) = -\operatorname{cosec}\left(\frac{\pi}{10}\right)\sec\left(\frac{\pi}{5}\right) = -4 \in Q$$

(c) $$1 - 2\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right) = 1 - \frac{1}{4} = \frac{3}{4} \in Q$$

(d) $$\left(2\cos^2\frac{\pi}{9}\right)\left(2\cos^2\frac{2\pi}{9}\right)\left(2\cos^2\frac{4\pi}{9}\right) = \frac{1}{8} \in Q$$

39. (a, b, c) : $2^{\operatorname{cosec}^2 x}\sqrt{(y-1)^2+1} \le 2$

$\Rightarrow$ $\operatorname{cosec}^2 x = 1$ and $y = 1 \Rightarrow$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $y = 1$

40. (a, c) : Since, α and β satisfy $x\cos\theta + y\sin\theta = 2a$

$\Rightarrow$ $(x^2 + y^2)\cos^2\theta - 4ax\cos\theta + (4a^2 - y^2) = 0$

$$\cos\alpha + \cos\beta = \frac{4ax}{x^2+y^2}, \cos\alpha\cdot\cos\beta = \frac{4a^2 - y^2}{x^2+y^2}$$

$$2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right) = 1 \Rightarrow 4\sin^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\beta}{2}\right) = 1$$

$\Rightarrow$ $\cos\alpha + \cos\beta = \cos\alpha \cdot \cos\beta$

41. (a, b, c, d) : $R \ge 2r \Rightarrow R^2 \ge \frac{4\Delta R}{2S} \Rightarrow R^2 \ge \frac{abc}{a+b+c}$

If $\angle C = 90° \Rightarrow a^2 + b^2 = c^2$, $c = 2R$

$r + 2R = (s - c)\tan\frac{C}{2} + c = s - c + c = s$

42. (b) : $\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}$

are the roots of $x^3 - 4x^2 - 4x + 8 = 0$

$$\sec\left(\frac{\pi}{7}\right) + \sec\left(\frac{3\pi}{7}\right) + \sec\left(\frac{5\pi}{7}\right) = 4$$

43. (b) : $8x^3 - 4x^2 - 4x + 1$

$$= 8\left(x - \cos\frac{\pi}{7}\right)\left(x - \cos\frac{3\pi}{7}\right)\left(x - \cos\frac{5\pi}{7}\right) \quad \text{...(i)}$$

Put $x = 1 \Rightarrow$ $\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right) = \frac{1}{8}$

44. (d) : Put $x = -1$ in (i), we get

$$\Rightarrow \cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right) = \frac{\sqrt{7}}{8}$$

(45 - 47) :

45. (a) **46. (b)** **47. (d)**

$\sin\alpha + \sin\beta = 1/4$ ...(i)

$$\Rightarrow 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 1/4 \quad \text{...(ii)}$$

$\cos\alpha + \cos\beta = 1/3$ ...(iii)

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 1/3 \quad \text{...(iv)}$$

Dividing (iv) by (ii), we have

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{3}{4}$$

$$\Rightarrow \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\times\frac{3}{4}}{1+\left(\frac{3}{4}\right)^2} = \frac{24}{25}$$

And $$\cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{1-\left(\frac{3}{4}\right)^2}{1+\left(\frac{3}{4}\right)^2} = \frac{7}{25}$$

48. A $\to$ R; B $\to$ S; C $\to$ P; D $\to$ Q

(A) $A = 20° \Rightarrow 3A = 60° \Rightarrow \cos 3A = \frac{1}{2}$

$\Rightarrow$ $8x^3 - 6x - 1 = 0$ where $x = \cos 20°$

(B) $A = 10° \Rightarrow \sin 3A = 1/2$

$\Rightarrow 8x^3 - 6x + 1 = 0$ where $x = \sin 10°$

(C) $A = 15° \Rightarrow \tan 3A = 45°$

$\Rightarrow x^3 - 3x^2 - 3x + 1 = 0$ where $x = \tan 15°$

(D) $A = 6° \Rightarrow \sin 5A = 1/2$

$\Rightarrow 32x^5 - 40x^3 + 10x - 1 = 0$ where $x = \sin 6°$

49. A → S; B → R; C → Q; D → P

(A) $\cos(2A + \theta) + \cos(2B + \theta)$

$= 2\cos(A + B + \theta)\cos(A - B) \le 2\cos(A - B)$

(B) $\cos 2A + \cos 2B = 2\cos(A + B)\cos(A - B)$

$\le 2\cos(A + B)$

(C) $y = \sec x$ always concave up

$\therefore \dfrac{\sec 2A + \sec 2B}{2} \ge \sec(A + B)$

(D) $\sqrt{\tan\theta + \cot\theta - 2\cos(2A + 2B)}$

$= \sqrt{(\sqrt{\tan\theta} + \sqrt{\cot\theta})^2 + 4\sin^2(A + B)} \ge 2\sin(A + B)$

50. A → Q; B → S; C → P

(A) We have, $\Delta = a^2 - (b - c)^2$

$\Rightarrow \Delta = a^2 - b^2 - c^2 + 2bc$

$\Rightarrow b^2 + c^2 - a^2 = 2bc - \Delta$

$\Rightarrow 2bc\cos A = 2bc - \dfrac{1}{2}bc\sin A \Rightarrow 4\cos A + \sin A = 4$

$\Rightarrow 4\left(1 - 2\sin^2\dfrac{A}{2}\right) + 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = 4$

$\Rightarrow \tan\dfrac{A}{2} = \dfrac{1}{4} \Rightarrow \tan A = \dfrac{8}{15}$

(B) Let $\tan A = 3k$, $\tan B = 4k$, $\tan C = 5k$

$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow 12k = 60k^3 \Rightarrow k = \dfrac{1}{\sqrt{5}}$

$\therefore \tan A = \dfrac{3}{\sqrt{5}}, \tan B = \dfrac{4}{\sqrt{5}}, \tan C = \sqrt{5}$

$\therefore \sin A \sin B \sin C = \dfrac{2\sqrt{5}}{7}$

(C) We have, $f(\theta) = \dfrac{2\cos^2\theta - 2\sin\theta\cos\theta}{2(\cos^2\theta - \sin^2\theta)} = \dfrac{1}{1 + \tan\theta}$

$\because \tan(\alpha + \beta) = 1$

$\Rightarrow f(\alpha) = \dfrac{1}{1 + \tan\alpha}, f(\beta) = \dfrac{1}{1 + \tan\beta}$

$\therefore f(\alpha)\, f(\beta) = \dfrac{1}{2}$

51. (3) : Let $\tan x = 2t$, $\tan y = 3t$, $\tan z = 5t$

$\sum \tan x = (\tan x \tan y \tan z) \Rightarrow t^2 = \dfrac{1}{3}$

$\tan^2 x + \tan^2 y + \tan^2 z = t^2(4 + 9 + 25) = 38t^2 \Rightarrow K = 3$

52. (1) : $4x^2 - 16x + 15 < 0$

$4x^2 - 10x - 6x + 15 < 0$

$2x(2x - 5) - 3(2x - 5) < 0$

$\Rightarrow \dfrac{3}{2} < x < \dfrac{5}{2} \Rightarrow x = 2$

$\therefore \tan\alpha = 2, \cos\beta = 1$

$\Rightarrow \dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{2 + 0}{2 - 0} = 1$

53. (4) : We have, $\sin\theta(1 + \sin^2\theta) = 1 - \sin^2\theta$

$\Rightarrow \sin\theta(2 - \cos^2\theta) = \cos^2\theta$

Squaring both sides, we get

$\sin^2\theta(2 - \cos^2\theta)^2 = \cos^4\theta$

$\Rightarrow (1 - \cos^2\theta)(4 - 4\cos^2\theta + \cos^4\theta) = \cos^4\theta$

$\Rightarrow -\cos^6\theta + 5\cos^4\theta - 8\cos^2\theta + 4 = \cos^4\theta$

$\therefore \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

54. (6) : $\cos(A - B) = \dfrac{1 - \tan^2\left(\dfrac{A - B}{2}\right)}{1 + \tan^2\left(\dfrac{A - B}{2}\right)}$

$\Rightarrow \tan\left(\dfrac{A - B}{2}\right) = \dfrac{1}{\sqrt{63}}$

Use Napier's analogy, we will get $\cot\dfrac{C}{2} = \dfrac{9}{\sqrt{63}}$

Then $\cos C = \dfrac{1}{8}, c = 6$

55. (2) : $r_1 = \dfrac{\Delta}{s - a}, r_2 = \dfrac{\Delta}{s - b}, r_3 = \dfrac{\Delta}{s - c}$

Substitute this value in given equation and take abc common

$\text{L.H.S} = \dfrac{abc}{\Delta}\left[\dfrac{s - a}{a} + \dfrac{s - b}{b} + \dfrac{s - c}{c}\right] = \dfrac{abc}{\Delta}\left[\sum\dfrac{s}{a} - 3\right]$

$\Rightarrow k = 2$

BRAIN @ WORK

STRAIGHT LINES

This article is a collection of shortcut methods, important formulas and MCQs along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PETs.

DEFINITION OF A STRAIGHT LINE

- A straight line is a curve such that every point on the line segment joining any two points on it lies on it.
- Every first degree equation in x, y represents a straight line.
- A first degree equation in x, y *i.e.*, $ax + by + c = 0$ represents a line, it means that all points (x, y) satisfying $ax + by + c = 0$ lie along a line. Thus, a line is also defined as the locus of a point satisfying the conditions $ax + by + c = 0$, where a, b, c are constants.

ANGLE OF INCLINATION OF LINE

The angle of inclination of a line which crosses the x-axis is the smallest angle θ which the line makes with the positive direction of x-axis in anticlockwise direction.

Note :

(i) When two lines are parallel, they have the same inclination.

(ii) The inclination of a line which is parallel to x-axis or coinciding with x-axis is $0°$.

(iii) If θ is the angle of inclination of the line, then $0 \le \theta < \pi$.

SLOPE OF A LINE

If a line makes an angle θ ($0 \le \theta < \pi$ and $\theta \ne \pi/2$) with the positive direction of x-axis, then $\tan\theta$ is called the slope or gradient of the line. Slope of a line is usually denoted by m.

- Slope of the line $ax + by + c = 0$ is $-\dfrac{a}{b} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y}$
- Slope of the line joining the point $A\ (x_1, y_1)$ and $B(x_2, y_2)$ is $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{\text{Difference of } y \text{ coordinates}}{\text{Difference of } x \text{ coordinates}}$

ANGLE BETWEEN TWO LINES

- The angle θ between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1m_2}\right|$,
- Two lines are parallel, if $m_1 = m_2$.
- Two lines are perpendicular, if $m_1m_2 = -1$.

INTERCEPT OF A LINE ON AXES

- If a line cuts x-axis at $(a, 0)$, then a is called the intercept of the line on x-axis.
- If a line cuts y-axis at $(0, b)$, then b is called the intercept of the line on y-axis.

Note :

(i) If a line is making equal intercept on the axes, then its slope is -1.

(ii) If a line is making equal length of intercept on the axes, then its slope is ± 1.

EQUATION OF LINES PARALLEL TO AXES

(a) Equation of x-axis is $y = 0$

Equation of y-axis is $x = 0$

(b) Equation of a line parallel to x-axis is $y = c$ (constant)

Equation of the line parallel to x-axis and passing through (α, β) is $y = \beta$.

(c) Equation of a line parallel to y-axis is $x = c$.

Equation of the line parallel to y-axis and passing through (α, β) is $x = \alpha$.

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EQUATION OF STRAIGHT LINE IN DIFFERENT STANDARD FORMS

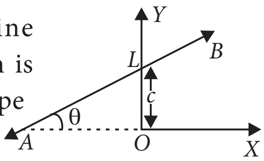
(a) General Equation :

Any equation of the form $ax + by + c = 0$, where a, b, c are constants and a, b are not simultaneously zero (*i.e.*, $a^2 + b^2 \neq 0$), always represents a straight line.

(b) Slope-Intercept Form:

If $OL = c$, $\angle BAX = \theta$, Equation of line AB is $y = mx + c$, where $m = \tan\theta$ = slope of AB. If L lies on the negative side of OY then c is taken as negative.

Note :



(i) The equation of a line passing through the origin is $y = mx$, where m is the slope of the line.

(ii) If the line is parallel to x-axis, then $m = 0$.

(c) Point Slope Form:

The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is $y - y_1 = m(x - x_1)$

(d) Two Point Form:

Equation of the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

(i) $y - y_1 = \dfrac{y_1 - y_2}{x_1 - x_2}(x - x_1)$ or $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

(ii) $y - y_2 = \dfrac{y_1 - y_2}{x_1 - x_2}(x - x_2)$ or $\dfrac{y - y_2}{y_1 - y_2} = \dfrac{x - x_2}{x_1 - x_2}$

(e) Determinant Form:

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) can also be written in the determinant form as $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(f) Intercept Form:

The equation of a line which cuts off intercepts a and b from the x-axis and y-axis respectively, is

$\dfrac{x}{a} + \dfrac{y}{b} = 1, (ab \neq 0)$

RELATION BETWEEN TWO LINES

Two lines given by the equation $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are (where $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$)

(i) Intersecting, consistent and having unique solution if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ (slopes are unequal)

(ii) Parallel and not coincident, inconsistent and having no solution, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ (slopes are equal)

(iii) Perpendicular, consistent and having unique solution, if $a_1a_2 + b_1b_2 = 0$

(iv) Identical or coincident, consistent having infinitely many solutions, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE

- The general form of equation of lines parallel to line $ax + by + c = 0$ is $ax + by + k = 0$
- Equation of the line parallel to the line $ax + by + c = 0$ and passing through (α, β) is $ax + by - (a\alpha + b\beta) = 0$
- The general form for the equation of lines perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$
- Equation of the line perpendicular to $ax + by + c = 0$ and passing through (α, β) is $bx - ay - (b\alpha - a\beta) = 0$

DISTANCE BETWEEN TWO PARALLEL LINES

Distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $d = \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

DISTANCE OF A POINT

- Distance (d) of a point $P(x_1, y_1)$ (not lying on the line) from the line $L : ax + by + c = 0$ is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

- Distance of line $L : ax + by + c = 0$ from the origin (0, 0) is

$$d = \frac{|c|}{\sqrt{a^2 + b^2}}$$

EQUATIONS OF STRAIGHT LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A GIVEN LINE

The equation of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

CONCURRENT LINES

The three lines $a_i x + b_i y + c_i = 0$, $i = 1, 2, 3$ are concurrent,

if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

The equation of lines passing through the intersection of lines $ax + by + c = 0$, $a'x + b'y + c' = 0$ must be of the form $ax + by + c + \lambda(a'x + b'y + c') = 0$, where a, b, c, a', b', c' are constant and λ is a parameter.

POSITION OF A GIVEN POINT RELATED TO A GIVEN LINE

Let the given line $L(x, y) \equiv ax + by + c = 0$ (where $b \neq 0$) and the given point is $P(x_1, y_1)$

(i) P lies above the line $ax + by + c = 0$, if $\frac{L(x_1, y_1)}{b} > 0$

(ii) P lies below the line $ax + by + c = 0$, if $\frac{L(x_1, y_1)}{b} < 0$

(iii) P lies on the line $ax + by + c = 0$, if $L(x_1, y_1) = 0$

POSITION OF TWO POINTS (x_1, y_1) AND (x_2, y_2) W.R.T. TO A GIVEN LINE

Let the given line be $L \equiv ax + by + c = 0$ and $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two given points.

(i) If $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ both are of the same sign or $\frac{L(x_1, y_1)}{L(x_2, y_2)} > 0$ or $L(x_1, y_1) \cdot L(x_2, y_2) > 0$

Then the point $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on same side of line $ax + by + c = 0$.

(ii) If $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ both are of opposite in sign or $\frac{L(x_1, y_1)}{L(x_2, y_2)} < 0$

or $L(x_1, y_1) \cdot L(x_2, y_2) < 0$

Then the point $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on opposite side of the line $ax + by + c = 0$.

(iii) (a) The side of the line where origin lies is known as origin side.

(b) A point (α, β) will lie on origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have same sign.

(c) A point (α, β) will lie on non-origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have opposite sign.

(iv) The point (x_1, y_1) lies in the angle between the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ in

(a) The acute angle, if

$$\frac{a_1a_2 + b_1b_2}{(a_1x_1 + b_1y_1 + c_1)(a_2x_1 + b_2y_1 + c_2)} < 0$$

(b) The obtuse angle, if

$$\frac{a_1a_2 + b_1b_2}{(a_1x_1 + b_1y_1 + c_1)(a_2x_1 + b_2y_1 + c_2)} > 0$$

(v) The origin lies in the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ in

(a) The acute angle, if $\frac{a_1a_2 + b_1b_2}{c_1c_2} < 0$

(b) The obtuse angle, if $\frac{a_1a_2 + b_1b_2}{c_1c_2} > 0$

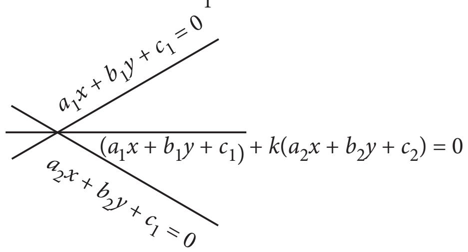
ANGLE BISECTOR

Consider two lines

$L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

Let $P(x, y)$ be any point on either of bisectors.

$\perp r$ Distance of P from $L_1 = \perp r$ Distance of P from L_2



$$\therefore \quad \frac{|a_1x + b_1y + c_1|}{\sqrt{(a_1^2 + b_1^2)}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{(a_2^2 + b_2^2)}}$$

PROBLEMS

Single Correct Answer Type

1. ABC is a variable triangle with the fixed vertex $C(1, 2)$ and A, B having the coordinates $(\cos t, \sin t)$, $(\sin t, -\cos t)$ respectively, where t is a parameter. The locus of the centroid of the ΔABC is

(a) $3(x^2 + y^2) - 2x - 4y - 1 = 0$
(b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
(c) $3(x^2 + y^2) + 2x + 4y - 1 = 0$
(d) $3(x^2 + y^2) + 2x + 4y + 1 = 0$

2. The lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ are the sides of a square. Equation of the remaining side of the square can be

(a) $2x - y - 14 = 0$ (b) $2x - y + 8 = 0$
(c) $2x - y - 10 = 0$ (d) $2x - y - 6 = 0$

3. If in a ΔABC (whose circumcentre is at the origin), and $a \leq \sin A$, then for any point (x, y) inside the circumcircle of ΔABC

(a) $|xy| < \frac{1}{8}$ (b) $|xy| > \frac{1}{8}$

(c) $\frac{1}{8} < xy < \frac{1}{2}$ (d) none of these

4. If the vertices of a triangle are $A(10, 4)$, $B(-4, 9)$ and $C(-2, -1)$, then the equation of its altitudes are

(a) $x - 5y + 10 = 0$, $12x + 5y + 3 = 0$ and $14x + 5y - 23 = 0$

(b) $14x - 5y + 10 = 0$, $x + 5y + 3 = 0$ and $14x - 5y + 23 = 0$

(c) $x - 5y + 1 = 0$, $12x + 5y + 3 = 0$ and $x - 5y + 23 = 0$

(d) $x - 5y + 10 = 0$, $12x + 5y + 3 = 0$ and $14x - 5y + 23 = 0$

5. If p_1 and p_2 are the lengths of the perpendiculars from the origin to the straight lines $x\sec\theta + y\,\mathrm{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, then the value of $4p_1^2 + p_2^2$ is

(a) $4a^2$ (b) $2a^2$

(c) a^2 (d) none of these

6. If (1, 1) and (–3, 5) are vertices of a diagonal of a square, then the equations of its sides through (1, 1) are

(a) $2x - y = 1$, $y - 1 = 0$ (b) $3x + y = 4$, $x - 1 = 0$

(c) $x = 1$, $y = 1$ (d) none of these

7. A family of lines is given by $(1 + 2\lambda)x + (1 - \lambda)y + \lambda = 0$, λ being the parameter. The line belonging to this family at the maximum distance from the point (1, 4) is

(a) $4x - y + 1 = 0$ (b) $33x + 12y + 7 = 0$

(c) $12x + 33y = 7$ (d) none of these

8. If $A(\sin\alpha, 1/\sqrt{2})$ and $B(1/\sqrt{2}, \cos\alpha)$, $-\pi \le \alpha \le \pi$, are two points on the same side of the line $x - y = 0$ then α belongs to the interval

(a) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (b) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(c) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (d) none of these

9. If the point $(\cos\theta, \sin\theta)$ does not fall in that angle between the lines $y = |x - 1|$ in which the origin lies, then θ belongs to

(a) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $(0, \pi)$ (d) none of these

10. If a ray travelling along the line $x = 1$ gets reflected from the line $x + y = 1$, then the equation of line along which the reflected ray travel is

(a) $y = 0$ (b) $x - y = 1$

(c) $x = 0$ (d) none of these

11. The point $P(2, 1)$ is shifted by $3\sqrt{2}$ parallel to the line $x + y = 1$, in the direction of increasing ordinate, to reach Q. The image of Q by the line $x + y = 1$ is

(a) (5, –2) (b) (–1, –2)

(c) (5, 4) (d) (–1, 4)

12. In triangle ABC; $A(1, 1)$, $B(4, -2)$, $C(5, 5)$. Equation of the internal angle bisector of $\angle A$ is

(a) $y = 1$ (b) $x - y = 1$

(c) $y = 5$ (d) $x + y = 5$

Multiple Correct Answer Type

13. A line which makes an acute angle θ with the positive direction of x-axis is drawn through the point $P(3, 4)$ to meet the line $x = 6$ at R and $y = 8$ at S, then

(a) $PR = 3\sec\theta$ (b) $PS = 4\,\mathrm{cosec}\theta$

(c) $PR + PS = \frac{2(3\sin\theta + 4\cos\theta)}{\sin 2\theta}$

(d) $\frac{9}{(PR)^2} + \frac{16}{(PS)^2} = 1$

14. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent $(a + b + c \ne 0)$, then

(a) $a^3 + b^3 + c^3 - 3abc = 0$

(b) $a = -b$ (c) $a = b = c$

(d) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

15. If $6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$, then the family of lines $ax + by + c = 0$ is concurrent at

(a) (–2, –3) (b) (3, –1)

(c) (2, 3) (d) (–3, 1)

16. If (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$, $5x - 6y - 1 = 0$, then

(a) $2\alpha + 3\alpha^2 - 1 > 0$ (b) $\alpha + 3\alpha^2 - 3 < 0$

(c) $\alpha + 2\alpha^2 - 3 < 0$ (d) $6\alpha^2 - 5\alpha + 1 > 0$

17. $A(1, 2)$ and $B(7, 10)$ are two points. If $P(x, y)$ is a point such that the angle APB is 60° and the area of the triangle APB is maximum, then which of the following is/are true?

(a) P lies on any line perpendicular to AB

(b) P lies on the right bisector of AB

(c) P lies on the straight line $3x + 4y = 36$

(d) P lies on the circle passing through the points (1, 2) and (7, 10) and having a radius of 10 units

18. If the points $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right), \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$, where $a \neq b \neq c \neq 1$, lies on the line $lx + my + n = 0$, then

(a) $a + b + c = -m/l$

(b) $ab + bc + ca = n/l$

(c) $abc = \frac{m+n}{l}$

(d) $abc - (bc + ca + ab) + 3(a + b + c) = 0$

Comprehension Type

Paragraph for Q. No. 19 to 21

Let $A(0, \beta)$, $B(-2, 0)$ and $C(1, 1)$ be the vertices of a triangle then

19. Angle A of the triangle ABC will be obtuse if β lies in

(a) $(-1, 2)$ (b) $\left(2, \frac{5}{2}\right)$

(c) $\left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right)$ (d) none of these

20. If I_1 is the interval of values of β for which A is obtuse and I_2 be the interval of values of β for which A is largest angle of ΔABC, then

(a) $I_1 = I_2$ (b) I_1 is a subset of I_2

(c) I_2 is a subset of I_1 (d) none of these

21. All the value of β for which angle A of the triangle ABC is largest lie in interval

(a) $(-2, 1)$ (b) $\left(-2, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right)$

(c) $\left(-2, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \sqrt{6}\right)$ (d) none of these

Paragraph for Q. No. 22 to 24

Given the equations of two sides of a square as $5x + 12y - 10 = 0$, $5x + 12y + 29 = 0$. Also given a point $M(-3, 5)$ lying on one of its sides.

22. The number of possible squares must be

(a) one (b) two

(c) four (d) none of these

23. The area of the square must be

(a) 9 sq. units (b) 6 sq. units

(c) 5 sq. units (d) none of these

24. If the possible equations of the remaining sides is $12x - 5y + \lambda = 0$, then λ cannot be

(a) 61 (b) 22

(c) 100 (d) 36

Paragraph for Q. No. 25 to 27

The vertex A of triangle ABC is $(3, -1)$. The equations of median BE and angular bisector CF are $6x + 10y - 59 = 0$ and $x - 4y + 10 = 0$. Then

25. Slope of the side BC must be

(a) 1/9 (b) $-2/9$

(c) 1/7 (d) none of these

26. The equation of AB must be

(a) $x + y = 2$ (b) $x + 4y = 0$

(c) $18x + 13y = 41$ (d) $23x - y = 70$

27. The length of the side AC must be

(a) $\sqrt{71}$ (b) $\sqrt{83}$

(c) $\sqrt{85}$ (d) none of these

Integer Answer Type

28. The number of integral values of m for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is

29. The line L has intercepts a and b on the coordinate axes. The coordinate axes are rotated through a fixed angle, keeping the origin fixed. If p and q are the intercepts of the line L on the new axes, then the value of $\dfrac{\frac{1}{a^2}+\frac{1}{b^2}}{\frac{1}{p^2}+\frac{1}{q^2}}$ is

30. If $\lambda : 1$ is the ratio in which the line joining the points $(2, 3)$ and $(4, 1)$ divides the segment joining the points $(1, 2)$ and $(4, 3)$, then the value of λ is

31. If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c' = 0$ have perpendicular diagonals, then the value of $\frac{a^2+b^2}{a'^2+b'^2}$ is

32. The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then the minimum sum of two different integral value of a is

33. A ray of light along $x+\sqrt{3}y=\sqrt{3}$ gets reflected upon reaching x-axis and the equation of the reflected ray is $ax + by + c = 0$ then the value of bc is

Matrix-Match Type

34. Match the following:

	Column-1		Column-II
(A)	If $P(1, 1)$, $Q(4, 2)$ and $R(x, 0)$ be three points such that $PR + RQ$ is minimum, then x is equal to	(P)	0
(B)	The area bound by the curves $\max\{\|x\|, \|y\|\} = 1$ is equal to	(Q)	1
(C)	The number of circles that touch all the three lines $2x - y = 5, x + y = 3$ and $4x - 2y = 7$ is equal to	(R)	2
(D)	If the point (a, a) lies between the lines $\|x + y\| = 6$ then $[\|a\|]$ can be equal to, ([·] represents the integral part)	(S)	3
		(T)	4

35. Match the following:

	Column-1		Column-II
(A)	If the point $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) internally, then	(P)	$-\frac{4\sqrt{2}}{3} < t < \frac{5\sqrt{2}}{6}$
(B)	Set of values of 't' for which point $P(t, t^2 - 2)$ lies inside the triangle formed by the lines $x + y = 1$, $y = x + 1$ and $y = -1$ is	(Q)	$1 < \|t\| < \frac{\sqrt{13}-1}{2}$
(C)	If $P(1+(t/\sqrt{2}), 2+(t/\sqrt{2}))$ be any point on a line, then the value of t for which the point P lies between parallel lines $x + 2y = 1$ and $2x + 4y = 15$, is	(R)	$0 < t < 1$
(D)	If the point $(1, t)$ always remains in the interior of the triangle formed by the lines $y = x$, $y = 0$ and $x + y = 4$, then	(S)	$\frac{4\sqrt{5}}{3} < t < \frac{5\sqrt{2}}{6}$

SOLUTIONS

1. (b) : Let $G(\alpha, \beta)$ be the centroid in any position.

Then $(\alpha, \beta) = \left(\frac{1+\cos t+\sin t}{3}, \frac{2+\sin t-\cos t}{3}\right)$

$\therefore \quad \alpha = \frac{1+\cos t+\sin t}{3}, \beta = \frac{2+\sin t-\cos t}{3}$

or $3\alpha - 1 = \cos t + \sin t$ .... (i)

and $3\beta - 2 = \sin t - \cos t$ ... (ii)

On squaring and adding (i) and (ii), we get

$(3\alpha - 1)^2 + (3\beta - 2)^2 = 2$

$\therefore$ The equation of the locus of the centroid is

$(3x - 1)^2 + (3y - 2)^2 = 2$

$\Rightarrow 3(x^2 + y^2) - 2x - 4y + 1 = 0$

2. (a) : Distance between $x + 2y + 3 = 0$ and $x + 2y - 7 = 0$ is $\frac{10}{\sqrt{5}}$

Let the remaining side be $2x - y + \lambda = 0$

We have, $\frac{|\lambda+4|}{\sqrt{5}} = \frac{10}{\sqrt{5}} \Rightarrow \lambda = 6, -14$

Thus remaining side is

$2x - y + 6 = 0$ or $2x - y - 14 = 0$

3. (a) : Since, $a \le \sin A \Rightarrow \frac{a}{\sin A} \le 1$

$\Rightarrow 2R \le 1 \Rightarrow R \le 1/2$

So, for any point (x, y) inside the circumcircle,

$x^2 + y^2 < \frac{1}{4}$

Using AM $\ge$ GM, $\left(\frac{x^2+y^2}{2} \ge |xy|\right) \Rightarrow |xy| < \frac{1}{8}$

4. (d) : Let AD, BE and CF be three altitudes of ΔABC.

Slope of $BC = -5 \Rightarrow$ Slope of $AD = 1/5$

Since AD passes through $A(10, 4)$.

$\therefore$ Equation of AD is $x - 5y + 10 = 0$

Similarly the equation of BE is $12x + 5y + 3 = 0$

and the equation of CF is $14x - 5y + 23 = 0$

5. (c) : We have, $p_1 = \frac{|-a|}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}}$

$\Rightarrow p_1^2 = \frac{a^2}{\sec^2\theta + \operatorname{cosec}^2\theta} = \frac{a^2 \sin^2\theta \cos^2\theta}{1}$

$\Rightarrow 4p_1^2 = a^2\sin^2 2\theta$ ...(i)

Also, $p_2 = \frac{|-a\cos 2\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |-a\cos 2\theta|$

$\Rightarrow p_2^2 = a^2\cos^2 2\theta$ ...(ii)

On adding (i) and (ii), we get $4p_1^2 + p_2^2 = a^2$

6. (c) : Since each side of a square makes 45° angle with its diagonals, so equations of the sides through (1, 1) are given by $y - 1 = \frac{-1 \mp \tan 45°}{1 \pm (-1)\tan 45°}(x - 1)$

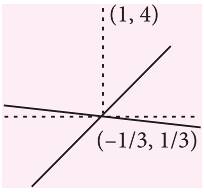
[$\because$ Slope of diagonal, $m = -1$]

$\Rightarrow y - 1 = \frac{-1 \mp 1}{1 \mp 1}(x - 1) \Rightarrow x - 1 = 0, y - 1 = 0.$

7. (c) : Here,

$x + y + \lambda(2x - y + 1) = 0$

$\Rightarrow$ Each line passes through the point of intersection of the lines $x + y = 0$ and $2x - y + 1 = 0$, which is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.



The required line passes through $\left(-\frac{1}{3}, \frac{1}{3}\right)$ and is perpendicular to the line joining (1, 4) and $\left(-\frac{1}{3}, \frac{1}{3}\right)$

$\therefore$ Slope of the required line $= -\frac{4}{11}$

$\therefore$ The required line is $y - \frac{1}{3} = -\frac{4}{11}\left(x + \frac{1}{3}\right)$

or $12x + 33y = 7$.

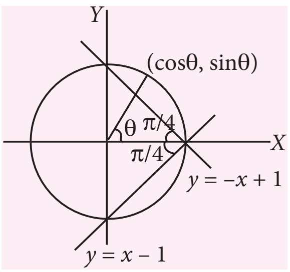
8. (a) : $\sin\alpha - \frac{1}{\sqrt{2}} > 0$ and $\frac{1}{\sqrt{2}} - \cos\alpha > 0$ ....(i)

or $\sin\alpha - \frac{1}{\sqrt{2}} < 0$ and $\frac{1}{\sqrt{2}} - \cos\alpha < 0$ .... (ii)

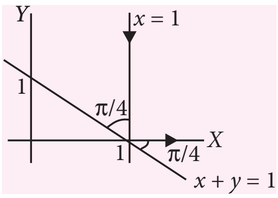
(i) $\Rightarrow$ $\sin\alpha > \frac{1}{\sqrt{2}}$ and $\cos\alpha < \frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4} < \alpha < \frac{3\pi}{4}$

(ii) $\Rightarrow$ $\sin\alpha < \frac{1}{\sqrt{2}}$ and $\cos\alpha > \frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4} < \alpha < \frac{\pi}{4}$

9. (b) : The lines are $y = x - 1$ and $y = -x + 1$ and $(\cos\theta, \sin\theta)$ is any point on the circle $x^2 + y^2 = 1$, where centre = (0, 0) and radius = 1. Clearly from the figure, θ can vary from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.



10. (a) :



Clearly from the figure, the reflected ray moves along the x-axis.

11. (d) : $Q = (2 \pm 3\sqrt{2}\cos\theta, 1 \pm 3\sqrt{2}\sin\theta)$, where $\tan\theta = -1$

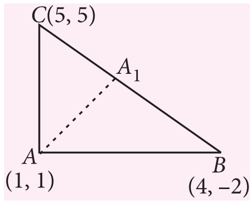
$$Q = \left(2 \pm 3\sqrt{2} \times \frac{-1}{\sqrt{2}}, 1 \pm 3\sqrt{2} \times \frac{1}{\sqrt{2}}\right) = (2 \mp 3,\ 1 \pm 3)$$

From the question, the y-coordinate of Q should be more than 1(= y-coordinate of P).

12. (a) : $m_{AB} = \frac{-2-1}{4-1} = -1$

and $m_{AC} = \frac{5-1}{5-1} = 1$

$\Rightarrow m_{AA_1} = 0$



Hence, equation of required bisector is, $y = 1$.

13. (a, b, c, d) : Equation of any line through $P(3, 4)$ making an angle θ with the positive direction of x-axis is $\frac{x-3}{\cos\theta} = \frac{y-4}{\sin\theta} = r$ ...(i)

where r is the distance of any point on the line from P. Therefore, coordinates of any point on the line (i) are $(3 + r\cos\theta, 4 + r\sin\theta)$ ...(ii)

If (ii) represents R, then

$3 + r\cos\theta = 6 \Rightarrow r = \frac{3}{\cos\theta} = PR$

If (ii) represents S, then

$4 + r\sin\theta = 8 \Rightarrow r = \frac{4}{\sin\theta} = PS$

Hence, $PR = 3\sec\theta$, $PS = 4\mathrm{cosec}\theta$

$$\Rightarrow PR + PS = \frac{3\sin\theta + 4\cos\theta}{\sin\theta\cos\theta} = \frac{2(3\sin\theta + 4\cos\theta)}{\sin 2\theta}$$

and $\left(\frac{3}{PR}\right)^2 + \left(\frac{4}{PS}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \frac{9}{(PR)^2} + \frac{16}{(PS)^2} = 1$$

14. (a, c, d)

15. (a, b) : Given, $6a^2 - 3b^2 - c^2 + 7ab - ac + 4bc = 0$

$\Rightarrow 6a^2 + (7b - c)a - (3b^2 - 4bc + c^2) = 0$

$$\Rightarrow a = \frac{c - 7b \pm \sqrt{(7b-c)^2 + 24(3b^2 - 4bc + c^2)}}{12}$$

$\Rightarrow 12a + 7b - c = \pm(11b - 5c)$

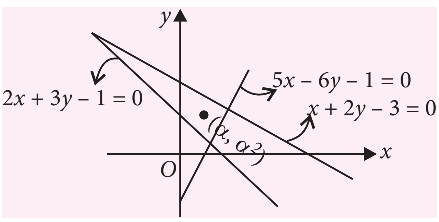
$\Rightarrow 3a - b + c = 0$ or $2a + 3b - c = 0$

$\Rightarrow (3, -1)$ or $(-2, -3)$ lies on line $ax + by + c = 0$

16. (a, c, d) : O and the point (α, α^2) lie to the opposite sides w.r.t. $2x + 3y - 1 = 0$

$\Rightarrow 2x + 3y - 1 = -1 < 0 \Rightarrow 2\alpha + 3\alpha^2 - 1 > 0$

O and the point (α, α^2) lie to the same side w.r.t $x + 2y - 3 = 0$

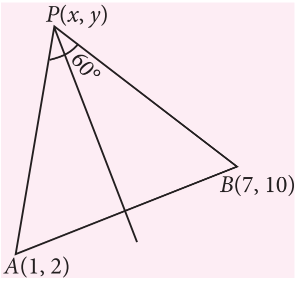


$$x + 2y - 3 = -3 < 0 \Rightarrow \alpha + 2\alpha^2 - 3 < 0$$

Again O and the point (α, α^2) lie to the same side w.r.t $5x - 6y - 1 = 0 \;\Rightarrow\; 5x - 6y - 1 = -1 < 0$

$\Rightarrow\; 5\alpha - 6\alpha^2 - 1 < 0 \quad \Rightarrow \quad 6\alpha^2 - 5\alpha + 1 > 0$

17. (b, c) : For the area to be maximum, P should lie on the right bisector of the side AB.



Coordinates of mid-point of AB are (4, 6)

Slope of $AB = \dfrac{8}{6} = \dfrac{4}{3}$

$\therefore$ Slope of perpendicular bisector $= -\dfrac{3}{4}$

Thus equation of line on which P lies is $3x + 4y = 36$

18. (a, b, d) : Since the given points lie on the line $lx + my + n = 0$ and a, b, c are the roots of the equation

$$l\left(\frac{t^3}{t-1}\right) + m\left(\frac{t^2-3}{t-1}\right) + n = 0$$

or $\; l\,t^3 + mt^2 + nt - (3m + n) = 0 \qquad$ ... (i)

$\Rightarrow\; a + b + c = -m/l\,;$

$\quad ab + bc + ca = n/l \qquad$ ... (ii)

and $abc = \dfrac{3m+n}{l} \qquad$ ... (iii)

So, that from (i), (ii) and (iii) we get

$$abc - (bc + ca + ab) + 3(a + b + c) = 0$$

19. (c) : Angle A will be obtuse if $BC^2 > AB^2 + AC^2$ and A, B, C are non-collinear.

$\Rightarrow\; 10 > \beta^2 + 4 + 1 + (\beta - 1)^2$

$\Rightarrow\; (\beta - 2)(\beta + 1) < 0 \;\Rightarrow \beta \in (-1, 2)$

But A, B, C are collinear for $\beta = 2/3$

$\Rightarrow$ Correct interval for β is $\left(-1, \dfrac{2}{3}\right) \cup \left(\dfrac{2}{3}, 2\right)$

20. (b) : Whenever A is obtuse, it is the largest angle also. But there may be more values of β for which A is largest but is not obtuse. (*i.e.*, right angle at A)

$\Rightarrow$ Choice (b) is correct.

21. (c) : Angle A will be largest if

$\quad a > b,\ a > c$ and A, B, C are non-collinear

$\Rightarrow\; a^2 > b^2,\ a^2 > c^2,\ \beta \neq 2/3$

$\Rightarrow\; 10 > 1 + (\beta - 1)^2,\ 10 > 4 + \beta^2$

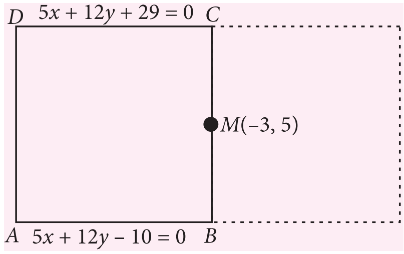
$\Rightarrow\; \beta^2 - 2\beta - 8 < 0,\ \beta^2 < 6$

$\Rightarrow\; -2 < \beta < 4,\ -\sqrt{6} < \beta < \sqrt{6}$

On taking intersection and rejecting $\beta = 2/3$, we get

$$\beta \in \left(-2, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \sqrt{6}\right)$$

(22-24) :



22. (b) : It is evident that $M(-3, 5)$ lies on some other side and possible squares are obviously two.

23. (a) : The perpendicular distance between AB and CD is $\dfrac{|29-(-10)|}{\sqrt{a^2+b^2}} = \dfrac{39}{13} = 3 \;\Rightarrow$ Area $= (3)^2 = 9$ sq. units

24. (d) : The equations of remaining sides can easily be determined as

$\quad 12x - 5y + 61 = 0,\ 12x - 5y + 22 = 0,$

$\quad 12x - 5y + 100 = 0$

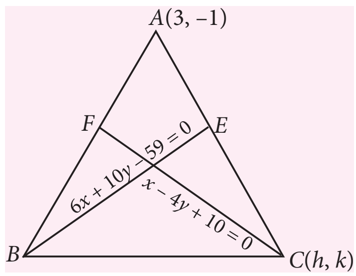
$\Rightarrow$ (d) is correct.

Equation of BC is $12x - 5y + 61 = 0$

$\Rightarrow$ Other possible sides are $12x - 5y + c = 0$, where

$$\frac{|c-61|}{13} = 3 \;\Rightarrow\; c = 100,\ c = 22$$

(25 - 27) :



25. (b) : Let coordinates of C be (h, k), then

$h - 4k + 10 = 0 \qquad$ ....(i)

and $E\left(\dfrac{h+3}{2}, \dfrac{k-1}{2}\right)$ lies on $6x + 10y - 59 = 0$

$$\Rightarrow\; 6\left(\frac{h+3}{2}\right) + 10\left(\frac{k-1}{2}\right) - 59 = 0$$

$\Rightarrow\; 3h + 5k = 55 \qquad$ ....(ii)

From (i) and (ii), we get $h = 10,\ k = 5$

$\Rightarrow$ Coordinates of C are (10, 5)

$\Rightarrow$ Slope of $AC = \dfrac{6}{7}$

$\Rightarrow$ Slope of BC must be given by

$$\left|\frac{m-\frac{1}{4}}{1+\frac{m}{4}}\right| = \left|\frac{\frac{6}{7}-\frac{1}{4}}{1+\frac{6}{7}\times\frac{1}{4}}\right| \;\Rightarrow\; m = \frac{6}{7}, \frac{-2}{9}$$

Since 6/7 represents slope, we have slope $BC = -\frac{2}{9}$

26. (c) : Equation of BC is $y-5=-\frac{2}{9}(x-10)$

or $2x + 9y - 65 = 0$

Now the point B can be found by solving BC and BE whence equation of AB can be determined as

$18x + 13y = 41$

27. (c)

28. (2) : $3x + 4(mx + 1) = 9$

$\Rightarrow (3+4m)x = 5 \Rightarrow x = \frac{5}{3+4m}$

For x to be an integer, $3 + 4m$ should be a divisor of 5 *i.e.*, 1, –1, 5 or –5.

$3 + 4m = 1 \Rightarrow m = -1/2$ (not an integer)

$3 + 4m = -1 \Rightarrow m = -1$ (integer)

$3 + 4m = 5 \Rightarrow m = 1/2$ (not an integer)

$3 + 4m = -5 \Rightarrow m = -2$ (integer)

Hence, there are two integral values of m.

29. (1) : The equation of the line L in the two coordinate systems are $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{X}{p}+\frac{Y}{q}=1$ where (X, Y) are the new coordinates of a point (x, y) when the axes are rotated through a fixed angle, keeping the origin fixed. As the length of the perpendicular from the origin has not changed,

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}} \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$$

$$\Rightarrow \frac{\frac{1}{a^2}+\frac{1}{b^2}}{\frac{1}{p^2}+\frac{1}{q^2}}=1$$

30. (1) : The equation of the line joining the points (2, 3) and (4, 1) is

$$y-3=\frac{1-3}{4-2}(x-2) \Rightarrow x + y - 5 = 0 \quad \text{....(i)}$$

Suppose the line joining (2, 3) and (4, 1) divides the segment joining (1, 2) and (4, 3) at the point P in the ratio of $\lambda : 1$.

Then the coordinates of P are $\left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+2}{\lambda+1}\right)$

Clearly, P lies on (i),

$$\Rightarrow \frac{4\lambda+1}{\lambda+1}+\frac{3\lambda+2}{\lambda+1}-5=0 \Rightarrow \lambda=1$$

31. (1) : Since the diagonals are perpendicular, so the given quadrilateral is a rhombus.

$\therefore$ Distance between two pairs of parallel sides are equal.

$$\Rightarrow \left|\frac{c'-c}{\sqrt{a^2+b^2}}\right|=\left|\frac{c'-c}{\sqrt{a'^2+b'^2}}\right| \Rightarrow a^2 + b^2 = a'^2 + b'^2$$

32. (3) : Given lines intersect at the point

$$P\left(\frac{1+|a|}{1+a}, \frac{a|a|-1}{1+a}\right)$$

This point will lie in first quadrant iff

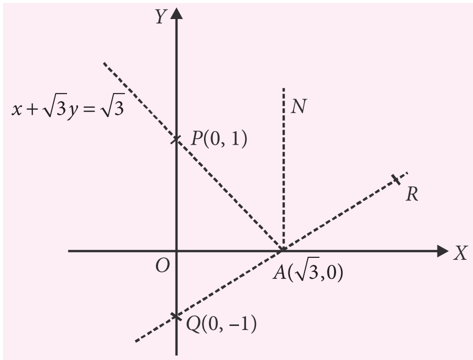
$1 + a > 0$ and $a\,|\,a\,| - 1 \geq 0$

$\Rightarrow a > -1$ and $a\,|\,a\,| - 1 \geq 0$

$\Rightarrow a^2 - 1 \geq 0$ if $a \geq 0$

or $-a^2 - 1 > 0$ if $0 < a < -1 \Rightarrow a \in [1, \infty)$

33. (3) : The line $x+\sqrt{3}y=\sqrt{3}$ cuts y-axis at $P(0, 1)$. Clearly its image $Q(0, -1)$ lies on the reflected ray AR produced backward.

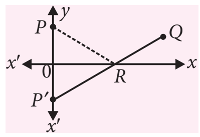


So, the equation of the reflected ray AR is

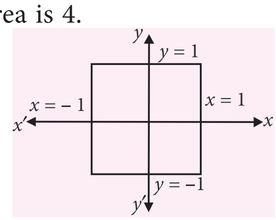
$$y + 1 = \frac{0+1}{\sqrt{3}-0}(x - 0) \text{ or } \sqrt{3}y = x-\sqrt{3}$$

34. (A) – (R); (B) – (T) ; (C) – (R) ; (D) – (P,Q,R)

(A) The image of P in the x-axis is $P'(1, -1)$. Equation of $P'Q$ is $y + 1 = \frac{3}{3}(x - 1)$, which intersects x–axis at (2, 0).



(B) The region is a square of side length 2, so the desired area is 4.



(C) As two of the lines are parallel, so two such circles are possible.

(D) The line $x = y$ cuts the lines

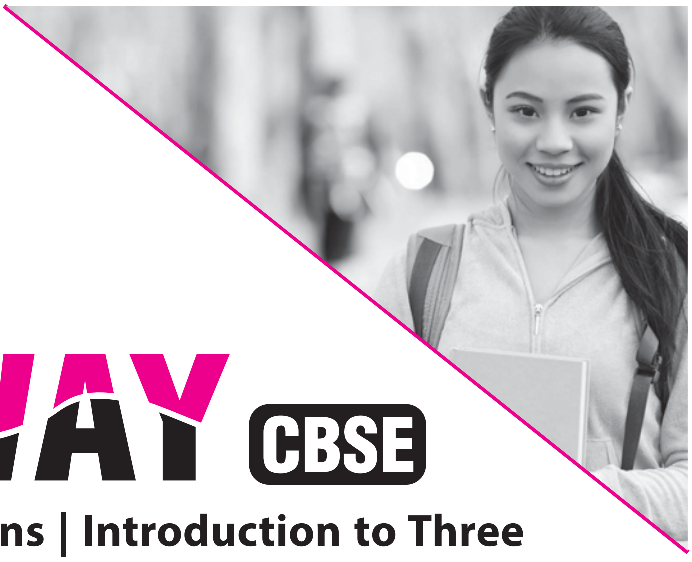
$x + y = \pm\ 6$ at (–3, –3) and (3, 3) $\Rightarrow -3 < a < 3$.

$\therefore$ $[|\,a\,|] = 0, 1, 2$.

35. (A) – (R) ; (B) – (Q) ; (C) – (P) ; (D) – (R)

CLASS XI | **Series 7**

ACE YOUR WAY CBSE



Straight Lines | Conic Sections | Introduction to Three Dimensional Geometry

HIGHLIGHTS

STRAIGHT LINES

DEFINITION

A path traced by a point in a constant direction and endlessly in its opposite direction is called straight line.

Distance Formula		Distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Section Formula	**Internal Division**	Coordinates of M, if M divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$ is $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$
	External Division	Coordinates of M, if M divides the join the of $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ is $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$
Mid Point Formula		Coordinates of M, if M bisects the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Area of a Triangle		Area of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is $\Delta = \frac{1}{2}\left\|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right\|$

SLOPE OF A LINE

If a line makes an angle θ with the positive direction of x-axis, then $\tan\theta$ is called the slope or gradient of the line. It is denoted by m.

Slope of a line	**Value of m**
Parallel to x-axis	0
Parallel to y-axis	not defined
Passing through two points (x_1, y_1) and (x_2, y_2)	$(y_2 - y_1)/(x_2 - x_1)$

CONDITION OF PARALLELISM AND PERPENDICULARITY

Two lines having slopes m_1 and m_2 are said to be

(i) Parallel iff $m_1 = m_2$

(ii) Perpendicular iff $m_1 m_2 = -1$

ANGLE BETWEEN TWO LINES

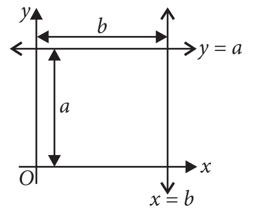
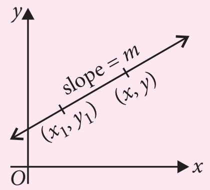
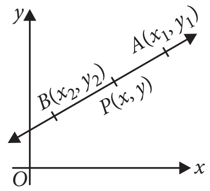
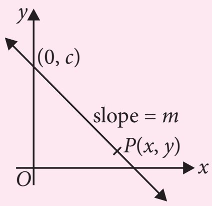
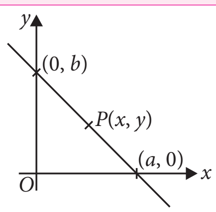
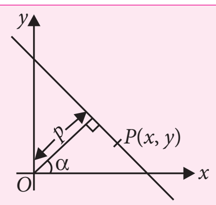
Angle between two lines having slopes m_1 and m_2 is,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$$

COLLINEARITY OF THREE POINTS

Three points A, B, C are said to be collinear, if slope of AB = slope of BC

DIFFERENT FORMS OF EQUATION OF LINE

	Equation	Figure
Equation of x-axis	$y = 0$	
Equation of y-axis	$x = 0$	
Equation of line parallel to x-axis	$y = a$	
Equation of line parallel to y-axis	$x = b$	
Point slope form : Equation of line whose slope is m and passing through the point (x_1, y_1)	$y - y_1 = m(x - x_1)$	
Two point form : Equation of line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$	$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	
Slope intercept form : Equation of line having slope m and cuts an intercept c on y-axis.	$y = mx + c$	
Intercept form : Equation of line cuts intercepts a and b on x and y-axes respectively.	$\frac{x}{a} + \frac{y}{b} = 1$	
Normal form : Equation of line having normal distance from origin p and this normal makes an angle α with the +ve x-axis.	$x\cos\alpha + y\sin\alpha = p$	

Note : If a line with slope m makes x-intercept d. Then equation of the line is $y = m(x - d)$.

GENERAL EQUATION OF A LINE

An equation of the form $Ax + By + C = 0$, where A, B, C are constants and A, B are not simultaneously zero is called the general equation of a line.

Different forms of $Ax + By + C = 0$

- **Slope - intercept form :** $y = -\frac{A}{B}x - \frac{C}{B}, B \neq 0$
- **Intercept form :** $\frac{x}{-C/A} + \frac{y}{-C/B} = 1, C \neq 0$
- **Normal form :** $x\cos\alpha + y\sin\alpha = p$

where $\cos\alpha = \pm\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \pm\frac{B}{\sqrt{A^2+B^2}}$

$$p = \pm\frac{C}{\sqrt{A^2+B^2}}$$

Distance of a point from a line : Distance of a point (x_1, y_1) from a line $Ax + By + C = 0$ is,

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$$

Distance between two parallel lines : Distance between two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is, $d = \frac{|C_2 - C_1|}{\sqrt{A^2+B^2}}$

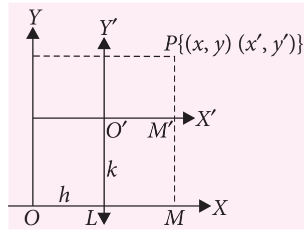
EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES

Let the two intersecting lines be $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$

So, equation of line passing through intersection of above lines is $A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0$ where λ is an arbitrary constant called parameter.

SHIFTING OF ORIGIN

Translation of axes



An equation corresponding to a set of points with reference to a system of coordinate axes may be simplified by taking the set of points in some other suitable coordinate system such that all geometric properties remain unchanged. We can form a transformation in which the new axes can be transformed parallel to the original axes and origin can be shifted to a new point. This kind of transformation is called a translation of axes. The coordinates of each point of the plane are changed under a translation of axes. To see how coordinates of a point of the plane changed under a translation of axes, let us take a point $P(x, y)$ referred to the axes OX and OY. Let $O'X'$ and $O'Y'$ be new axes parallel to OX and OY respectively, where O' is the new origin. Let (h, k) be the coordinates of O' referred to the old axes, *i.e.*, $OL = h$ and $LO' = k$. Also, $OM = x$ and $MP = y$

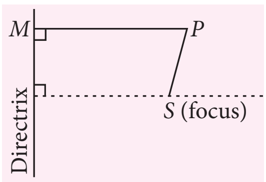
The transformation relation between new coordinates (x', y') and old coordinates (x, y) are given by

$$x = x' + h, y = y' + k$$

CONIC SECTIONS

DEFINITION

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant, is known as a conic section or a conic.



The fixed point is called focus of the conic, fixed line is called directrix of the conic and the constant ratio is called eccentricity of the conic.

CIRCLE, PARABOLA, ELLIPSE AND HYPERBOLA

When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations.

Relationship/ Value of angles	Conic	
$\beta = 90°$	Circle	α : Semi vertical angle of right circular cone. β : Angle made by the intersecting plane with the vertical axis of the cone.
$\alpha < \beta < 90°$	Ellipse	
$\beta = \alpha$	Parabola	
$0 \le \beta < \alpha$	Hyperbola	

Degenerated Conic Section :

When the plane cuts at the vertex of the cone, we have the following situations.

Relationship/ Value of angles	Conic	Degenerated conic	α : Semi vertical angle of right circular cone. β : Angle made by the intersecting plane with the vertical axis of the cone.
$\beta < \beta \le 90°$	Point		
$\beta = \alpha$	Straight line	Parabola	
$0 \le \beta < \alpha$	Pair of intersecting straight lines	Hyperbola	

CIRCLE

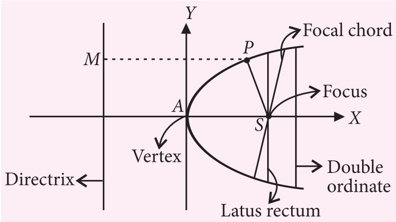
A circle is the locus of a point which moves in a plane so that its distance from a fixed point remains constant. The fixed point and the constant distance are called centre and radius of the circle respectively.

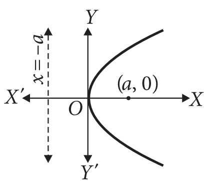
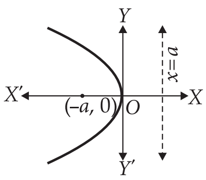
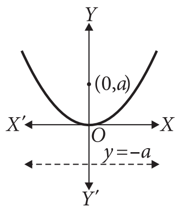
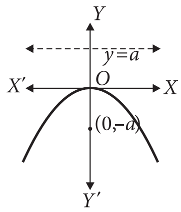
EQUATION OF CIRCLE

Equation of circle having centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

PARABOLA

A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point. The fixed line is called the directrix and fixed point is called focus of the parabola.

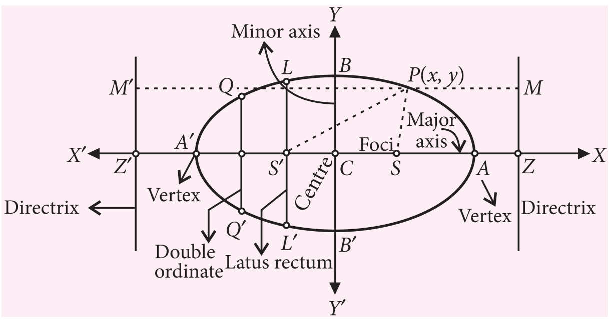


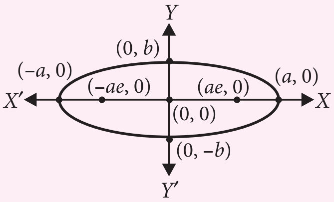
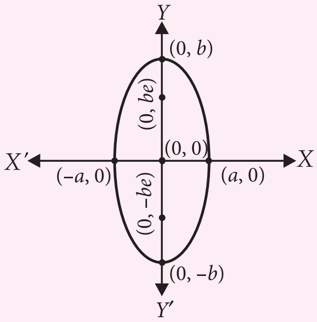
Standard equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Graph				
Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Coordinates of vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Extremities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus rectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$

Note : Parabola is symmetric with respect to the axis of the parabola. If the equation has y^2 term, then the axis of symmetry is along the x-axis and if the equation has x^2 term, then the axis of symmetry is along the y-axis.

ELLIPSE

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is constant. The two fixed points are called the foci of the ellipse.



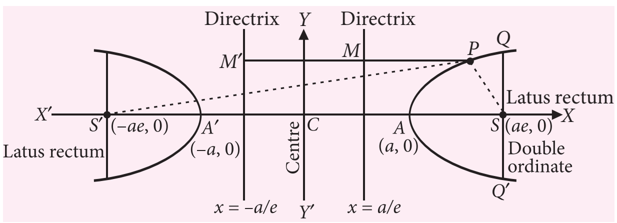
Fundamental terms	Horizontal ellipse	Vertical ellipse
Standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$
Graph		
Coordinates of the centre	(0, 0)	(0, 0)
Coordinates of the vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal distance or radii	$\lvert SP\rvert = (a-ex_1)$ and $\lvert S'P\rvert = (a+ex_1)$	$\lvert SP\rvert = (b-ey_1)$ and $\lvert S'P\rvert = (b+ey_1)$
Sum of focal radii = $\lvert SP\rvert + \lvert S'P\rvert$	$2a$	$2b$
Distance between foci	$2ae$	$2be$

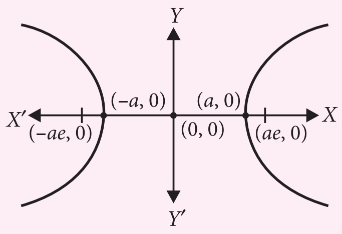
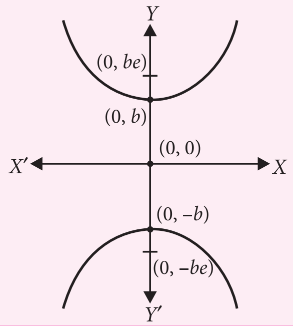
Note :

- If length of major axis is $2a$, length of minor axis is $2b$ and distance between the foci is $2c$, then $a^2 = b^2 + c^2$
- Ellipse is symmetric w.r.t. both the coordinate axes.
- The foci always lie on the major axis.
- If length of semi-major axis and semi-minor axis are equal, then the ellipse become circle.
- When distance between foci $(SS') = a$ then $b = 0$. The ellipse is reduced to the line segment SS'.

HYPERBOLA

A hyperbola is the set of all points in a plane, the difference of whose distance from two fixed points is constant.



Fundamental terms	Hyperbola	Conjugate Hyperbola
Standard equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$
Graph		
Coordinates of the centre	(0, 0)	(0, 0)
Coordinates of the vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm\left(\frac{a}{e}\right)$	$y=\pm\left(\frac{b}{e}\right)$
Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Difference of focal radii = $\lvert SP\rvert - \lvert S'P\rvert$	$2a$	$2b$
Distance between foci	$2ae$	$2be$

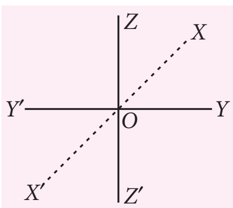
Note :

- A hyperbola in which $a = b$ is called an equilateral hyperbola.
- Hyperbola is symmetric w.r.t. both the axes.
- The foci are always on the transverse axis.

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

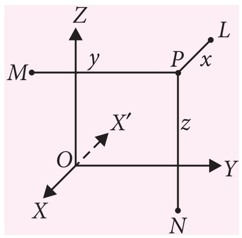
COORDINATE AXES AND COORDINATE PLANE

Let XOX', YOY' and ZOZ' be three mutually perpendicular lines intersecting at O. The lines XOX', YOY' and ZOZ' are called coordinate axes.Also the planes XOY, YOZ and ZOX are known as coordinate planes.



COORDINATES OF A POINT IN SPACE

Let P be any point in space. Draw PL, PM, PN perpendicular to the YZ, ZX and XY planes respectively, then



(i) LP is called the x-coordinate of P.

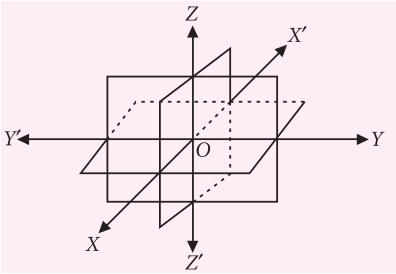
(ii) MP is called the y-coordinate of P.

(iii) NP is called the z-coordinate of P.

These three, taken together, are known as coordinates of P. Thus the coordinates of any point in space are the

perpendicular distances of the point from the YZ, ZX and XY planes respectively.

Signs of Coordinates in Eight Octants :



The octants $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ are denoted by I, II, III, ..., VIII respectively.

	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

DISTANCE BETWEEN TWO POINTS

Distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

SECTION FORMULA

Let M be a point which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$, then

Types of division	Coordinates of M
Internal	$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$
External	$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$

MID POINT FORMULA

Let M be the mid point of the line joining $A\ (x_1, y_1, z_1)$ and $B\ (x_2, y_2, z_2)$. Then coordinates of M is given by, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

PROBLEMS

Very Short Answer Type

1. Find the slope of the line joining the points $(-2, -5)$ and $(4, -6)$.
2. Find the equation of the circle whose centre is $(1, 2)$ and which passes through the point $(4, 6)$.
3. Find the point which is equidistant from points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
4. Find the equation of the line upon which the length of perpendicular p from origin and the angle α made by this perpendicular with the positive direction of x-axis are $p = 5$, $\alpha = 135°$.
5. Find the centre and radius of the circle $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

Short Answer Type

6. A line through the point $A(2, 0)$ which makes an angle of 30° with the positive direction of x-axis is rotated about A in clockwise direction through an angle 15°. Find the equation of the straight line in the new position.
7. Find the equation of a circle of radius 5 cm whose centre lies on x - axis and passes through the point $(2, 3)$.
8. Is the triangle, whose vertices are $(5, -6)$, $(1, 2)$ and $(-7, -2)$ a right angled triangle, an acute angled triangle or an obtuse angled triangle ?
9. Find the ratio in which the line joining the points $(4, 4, -10)$ and $(-2, 2, 4)$ is divided by the XY-plane.
10. Find the equation of the hyperbola whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16, taking transverse and conjugate axes of the hyperbola as x and y-axes respectively.

Long Answer Type - I

11. Find the equation of the lines which cut off intercepts on the axes whose sum and product are 1 and -6, respectively.
12. Find the equation of the ellipse with foci at $(\pm 5, 0)$ and $x = \frac{36}{5}$ as one directrix.
13. If P is any point on a hyperbola and N is the foot of the perpendicular from P on the transverse axis, then prove that $\frac{(PN)^2}{(AN)(A'N)} = \frac{b^2}{a^2}$.
14. Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of x-axis.
15. Prove that the equation of the parabola whose vertex and focus are on the x-axis at distances a' and a'' respectively from the origin is $y^2 = 4(a'' - a')(x - a')$.

Long Answer Type - II

16. Find the equations of the two lines which can be drawn through the point (2, 2) to make an angle of 45° with the line $x + y = 2$.

17. Find the equation of the straight line which passes through the point of intersection of lines $3x - 4y - 7 = 0$ and $12x - 5y - 13 = 0$ and is perpendicular to the line $2x - 3y + 5 = 0$.

18. Find the focus and the equation of the parabola whose vertex is (6, – 3) and directrix is $3x - 5y + 1 = 0$.

19. Find the length and equation of major and minor axes, centre, eccentricity, foci, equation of directrices, vertices and length of latus rectum of the ellipse $\frac{x^2}{225}+\frac{y^2}{289}=1$.

20. If S and S' be the foci, C the centre and P be any point on a rectangular hyperbola having lengths of transverse and conjugate axes equal, show that $SP \cdot S'P = CP^2$.

SOLUTIONS

1. Required slope $=\frac{-6-(-5)}{4-(-2)}=-\frac{1}{6}$

2. Centre of the given circle, $C \equiv (1, 2)$. Let $P \equiv (4, 6)$
Since point $P(4, 6)$ lies on the circle

$\therefore$ Radius of circle $= CP = \sqrt{(1-4)^2+(2-6)^2}=5$

Now, the equation of the circle is
$(x - 1)^2 + (y - 2)^2 = 5^2$ or $x^2 + y^2 - 2x - 4y - 20 = 0$

3. Let $P(x, y, z)$ be the required point
Given, $OP = AP \Rightarrow OP^2 = AP^2$

$\therefore\ x^2 + y^2 + z^2 = (x - a)^2 + y^2 + z^2 \Rightarrow x = \frac{a}{2}$

Similarly, $OP = BP \Rightarrow y = \frac{a}{2}$ and $OP = CP \Rightarrow z = \frac{c}{2}$

Thus, $P \equiv \left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

4. Here $p = 5$ and $\alpha = 135°$, therefore, required equation of the line is $x \cos 135° + y \sin 135° = 5$

$\Rightarrow\ -\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=5 \Rightarrow x-y+5\sqrt{2}=0$

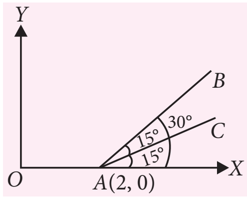
5. We have $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

$$\Rightarrow \left(x^2-\frac{8}{3}x\right)+\left(y^2-\frac{10}{3}y\right)=-1$$

$$\Rightarrow \left(x-\frac{4}{3}\right)^2+\left(y-\frac{5}{3}\right)^2=\left(\frac{4}{3}\sqrt{2}\right)^2$$

Hence, centre of the circle is $\left(\frac{4}{3},\frac{5}{3}\right)$ and radius is $\frac{4\sqrt{2}}{3}$ units.

6. Given, $A \equiv (2, 0)$, AB is the initial position of the line and AC is its new position.
Given, $\angle BAX = 30°$ and $\angle BAC = 15°$ $\therefore$ $\angle CAX = 15°$



Now slope of line AC
$= \tan 15° = 2 - \sqrt{3}$
Now equation of line AC will be

$y - 0 = (2-\sqrt{3})(x-2)$

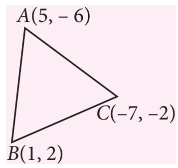
or $(2-\sqrt{3})x-y-4+2\sqrt{3}=0$

7. Let the coordinates of the centre of the required circle be $C(a, 0)$. Since it passes through $P(2, 3)$.
$\therefore$ CP = radius = 5 cm

$\Rightarrow \sqrt{(a-2)^2+(0-3)^2}=5 \Rightarrow a - 2 = \pm 4$

$\Rightarrow a = 6$ or, $a = -2$
Thus, the coordinates of the centre are (6, 0) or (–2, 0).
Hence, the equation of the required circle are
$(x - 6)^2 + (y - 0)^2 = 5^2$ and $(x + 2)^2 + (y - 0)^2 = 5^2$
$\Rightarrow x^2 + y^2 - 12x + 11 = 0$ and $x^2 + y^2 + 4x - 21 = 0$

8. Let $A \equiv (5, -6)$, $B \equiv (1, 2)$ and $C \equiv (-7, -2)$



Slope of $AB=\frac{-6-2}{5-1}=-2=m_1$ (say)

Slope of $BC=\frac{2+2}{1+7}=\frac{1}{2}=m_2$ (say)

Slope of $AC=\frac{-6+2}{5+7}=-\frac{1}{3}=m_3$ (say)

Since $m_1m_2 = -1$, hence $AB \perp BC$ $\therefore$ $\angle ABC = 90°$
Thus ΔABC is a right angled triangle.

9. Let $A \equiv (4, 4, -10)$, $B \equiv (-2, 2, 4)$
Let the line joining A and B be divided by the XY-plane at point P in the ratio $\lambda : 1$

Then, $P \equiv \left(\frac{-2\lambda+4}{\lambda+1},\frac{2\lambda+4}{\lambda+1},\frac{4\lambda-10}{\lambda+1}\right)$

Since P lies on the XY-plane, therefore z-coordinate of P will be zero.

$\therefore\ \frac{4\lambda-10}{\lambda+1}=0 \Rightarrow \lambda=\frac{5}{2}$

$\therefore$ Required ratio is 5 : 2 (internal).

10. Given, $2ae = 16 \Rightarrow 2 \cdot a\sqrt{2} = 16 \Rightarrow a = 4\sqrt{2}$
Also $b^2 = a^2(e^2 - 1) = 32(2 - 1) = 32$
Thus $a^2 = 32$ and $b^2 = 32$

$\therefore$ The required equation of hyperbola is

$$\frac{x^2}{32}-\frac{y^2}{32}=1 \text{ or } x^2 - y^2 = 32$$

11. Let a and b be the intercepts of the line on x and y axes respectively.

$\therefore$ Equation of the line will be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

Given, $a + b = 1$ ...(ii) and $ab = -6$ ...(iii)

Putting the value of b from (ii) in (iii), we get

$$a(1 - a) = -6 \Rightarrow a^2 - a - 6 = 0 \Rightarrow a = 3, -2$$

From (ii), when $a = 3$, $b = -2$ and when $a = -2$, $b = 3$

$\therefore$ Required equation of the lines are

$$\frac{x}{3}+\frac{y}{-2}=1 \text{ and } \frac{x}{-2}+\frac{y}{3}=1$$

i.e., $2x - 3y - 6 = 0$ and $3x - 2y + 6 = 0$

12. Let $S \equiv (5, 0)$ and $S' \equiv (-5, 0)$ be the foci.

Centre of the ellipse will be $C(0, 0)$

Clearly, foci S and S' lie on x-axis. Therefore, major axis of the ellipse will be parallel to x-axis.

Let a and b be the length of semi-major and semi-minor axes respectively of the ellipse.

Then, $ae = 5$ ...(i)

Also equation of one directrix is given to be $x=\frac{36}{5}$

$\therefore$ $\frac{a}{e}=\frac{36}{5}$ ...(ii)

On multiplying (i) and (ii), we get $a^2 = 36$ $\therefore$ $a = 6$

From (i), $e=\frac{5}{6}$

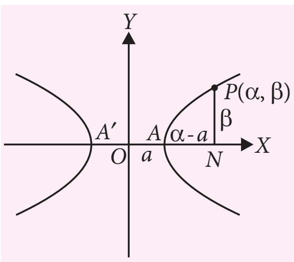
Now, $b^2 = a^2(1 - e^2) = 36\left(1-\frac{25}{36}\right)=11$

$\therefore$ The required equation of the ellipse is

$$\frac{x^2}{36}+\frac{y^2}{11}=1$$

13. Let $P(\alpha, \beta)$ be any point on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(i)

Then, $\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}=1$

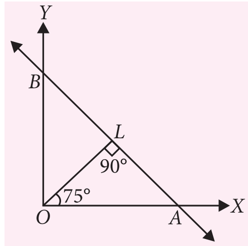


$$\Rightarrow \frac{\alpha^2}{a^2}-1=\frac{\beta^2}{b^2}$$

$$\Rightarrow \frac{(\alpha-a)(\alpha+a)}{a^2}=\frac{\beta^2}{b^2}$$

$$\Rightarrow \frac{(AN)(A'N)}{a^2}=\frac{(PN)^2}{b^2} \Rightarrow \frac{(PN)^2}{(AN)(A'N)}=\frac{b^2}{a^2}$$

14. Let AB be the required line and OL be perpendicular to it.



Given, $OL = 3\sqrt{2}$

and $\angle LOA = 75°$

$\therefore$ Equation of line AB will be

$x \cos 75° + y \sin 75° = 3\sqrt{2}$ (Normal form) ...(i)

Now, $\cos 75° = \cos(30° + 45°) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

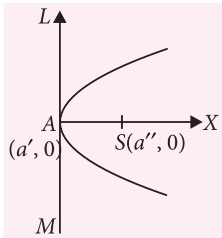
and similarly $\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore$ From (i), equation of line AB is

$$x\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+y\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=3\sqrt{2}$$

or $(\sqrt{3}-1)x+(\sqrt{3}+1)y-12=0$

15. Since the vertex and focus of the parabola are on x-axis, therefore, axis of the parabola is x-axis. Let A be the vertex and S the focus of the parabola. Since vertex and focus of the parabola are at distances a' and a'' from the origin, therefore



$$A \equiv (a', 0) \text{ and } S = (a'', 0)$$

We know that equation of a parabola whose vertex is (α, β) and axis is parallel to x-axis is

$$(y - \beta)^2 = 4a(x - \alpha)$$

where, a = distance between focus and vertex

Here $\alpha = a'$, $\beta = 0$ and $a = AS = a'' - a'$

$\therefore$ Required equation of the parabola will be $y^2 = 4(a'' - a')(x - a')$.

16. The given line is $x + y = 2$ ...(i)

Slope of this line $= -1$

Let the slope of the line through (2, 2), which makes an angle of 45° with line (i), be m, then

$$\tan 45° = \left|\frac{m-(-1)}{1+m(-1)}\right| \Rightarrow \pm 1 = \frac{m+1}{1-m}$$

$\Rightarrow \pm(1 - m) = m + 1$

Case I : $1 - m = m + 1 \Rightarrow 2m = 0 \Rightarrow m = 0$

Hence, one of the required lines through (2, 2) is $y - 2 = 0(x - 2)$ (taking $m = 0$)

$\Rightarrow y = 2$

Case II : $-(1 - m) = m + 1 \Rightarrow -1 + m = m + 1$

$\therefore$ Slope is not defined.

$\therefore$ The other required line ought to be vertical.

$\Rightarrow$ The equation of this line is $x - 2 = 0$

($\because$ It passes through the point (2, 2))

17. Given lines are

$3x - 4y - 7 = 0$ ...(i)

$12x - 5y - 13 = 0$ ...(ii)

$2x - 3y + 5 = 0$ ...(iii)

Equation of any line through the point of intersection of lines (i) and (ii) is

$3x - 4y - 7 + k(12x - 5y - 13) = 0$

or $(3 + 12k)\,x - (4 + 5k)\,y - (7 + 13k) = 0$ ...(iv)

Slope of line (iv) is $\frac{3+12k}{4+5k}$ and slope of line (iii) is $\frac{2}{3}$

If line (iv) is perpendicular to line (iii), then

$\left(\frac{3+12k}{4+5k}\right)\cdot\frac{2}{3} = -1$ or $6 + 24k = -12 - 15k$

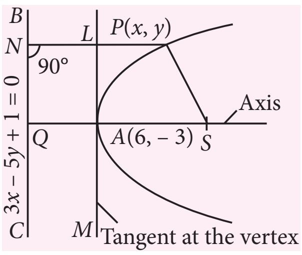
or $39k = -18$ $\therefore$ $k = -\frac{6}{13}$

Putting the value of k in (iv), the equation of required line is

$$\left(3-\frac{72}{13}\right)x-\left(4-\frac{30}{13}\right)y-(7-6)=0$$

or $33x + 22y + 13 = 0$

18. Let $A(6, -3)$ be the vertex and BC be the directrix of the parabola. Let S be the focus.



Equation of directrix BC is $3x - 5y + 1 = 0$ ...(i)

Equation of any line perpendicular to line (i) may be taken as $5x + 3y + k = 0$ ...(ii)

If line (ii) passes through $A(6, -3)$, then

$5 \times 6 + 3(-3) + k = 0$ or $k = -21$

$\therefore$ From (ii), equation of line AQ becomes

$5x + 3y - 21 = 0$ ...(iii)

Solving (i) and (iii) by cross multiplication method, we get

$$\frac{x}{105-3}=\frac{y}{5+63}=\frac{1}{9+25} \quad \therefore\ x = 3,\ y = 2$$

$\therefore$ $Q \equiv (3, 2)$. Given $A \equiv (6, -3)$

Let $S \equiv (\alpha, \beta)$. Since A is the mid-point of QS

$$\therefore\ 6=\frac{3+\alpha}{2}\Rightarrow\alpha=9 \text{ and } -3=\frac{2+\beta}{2}\Rightarrow\beta=-8$$

$\therefore$ $S \equiv (9, -8)$

Let $P(x, y)$ be an arbitrary point on the parabola, then $PS = PN$

$$\Rightarrow \sqrt{(x-9)^2+(y+8)^2}=\frac{|3x-5y+1|}{\sqrt{9+25}}$$

$\Rightarrow$ $25x^2 + 9y^2 - 618x + 554y + 30xy + 4929 = 0$

This is the required equation of the parabola.

19. Equation of ellipse is $\frac{x^2}{15^2}+\frac{y^2}{17^2}=1$

$a = 15$, $b = 17$ so, $a < b$

Length of major axis $= 2b = 34$

Length of minor axis $= 2a = 30$

Equation of major axis is $x = 0$

Equation of minor axis is $y = 0$

Coordinates of centre are (0, 0)

Eccentricity of the ellipse

$$e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{225}{289}}=\sqrt{\frac{64}{289}}=\frac{8}{17}$$

Coordinates of foci are given by

$(0, \pm be)$ *i.e.*, $(0, \pm 8)$

Equation of directrices are

$$y=\pm\frac{b}{e} \text{ or } y=\pm\frac{17\cdot17}{8}$$

or $8y \mp 289 = 0$

Coordinates of vertices are given by $(0, \pm b)$ *i.e.*, $(0, \pm 17)$

$$\text{Length of latus rectum }=\frac{2a^2}{b}=\frac{2\times225}{17}=\frac{450}{17}$$

20. Let the equation of the rectangular hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(i)$$

Given, $a = b$, therefore (i) becomes $x^2 - y^2 = a^2$ ...(ii)

$$\text{Now, } e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{a^2}{a^2}}=\sqrt{2}$$

Thus $e=\sqrt{2}, C\equiv(0,0), S\equiv(\sqrt{2}\,a,0), S'\equiv(-\sqrt{2}\,a,0)$

Let $P \equiv (\alpha, \beta)$. Since P lies on (ii)

$\therefore$ $\alpha^2 - \beta^2 = a^2$ ...(iii)

Now, $SP^2 = (\sqrt{2}\,a-\alpha)^2+\beta^2 = 2a^2+\alpha^2+\beta^2-2\sqrt{2}\,a\,\alpha$

and $S'P^2 = (-\sqrt{2}\,a-\alpha)^2+\beta^2 = 2a^2+\alpha^2+\beta^2+2\sqrt{2}\,a\,\alpha$

Now, $SP^2\cdot S'P^2 = (2a^2+\alpha^2+\beta^2)^2 - 8a^2\alpha^2$

$= 4a^4 + 4a^2(\alpha^2+\beta^2) + (\alpha^2+\beta^2)^2 - 8a^2\alpha^2$

$= 4a^2(a^2 - 2\alpha^2) + 4a^2(\alpha^2+\beta^2) + (\alpha^2+\beta^2)^2$

$= 4a^2(\alpha^2 - \beta^2 - 2\alpha^2) + 4a^2(\alpha^2+\beta^2) + (\alpha^2+\beta^2)^2$ [from (iii)]

$= (\alpha^2+\beta^2)^2 = (CP^2)^2 = CP^4$

$\therefore$ $SP\cdot S'P = CP^2$

◈◈

MPP-5 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

Sequences & Series

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If a_1, a_2 .... a_n are positive real numbers whose product is a fixed number c then the minimum value of $a_1 + a_2 + ... + a_{n-1} + 2a_n$ is

(a) $n(2c)^{\frac{1}{n}}$ (b) $(n+1)c^{\frac{1}{n}}$

(c) $2nc^{\frac{1}{n}}$ (d) $(n+1)(2c)^{\frac{1}{n}}$

2. The A.M. between m and n and the G.M. between a and b are each equal to $\frac{ma+nb}{m+n}$. Then m =

(a) $\frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (b) $\frac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$

(c) $\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (d) $\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$

3. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$, then a, b, c, d

(a) are in A.P. (b) are in G.P.

(c) are in H.P. (d) satisfy $ab = cd$

4. If $1\cdot 3 + 3\cdot 3^2 + 5\cdot 3^3 + 7\cdot 3^4 + ...$ upto n terms is equal to $3 + (n-1)\cdot 3^b$, then b =

(a) n (b) $n - 1$

(c) $2n - 1$ (d) $n + 1$

5. If x = 111....1 (20 digits), y = 333...3 (10 digits) and z = 222....2 (10 digits), then $\frac{x-y^2}{z}$ =

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) 3

6. Four geometric means are inserted between the numbers $2^{11} - 1$ and $2^{11} + 1$. The product of these geometric means is

(a) $2^{44} - 2^{23} + 1$ (b) $2^{44} - 2^{22} + 1$

(c) $2^{22} - 2^{11} + 1$ (d) $2^{22} - 2^{12} + 1$.

One or More Than One Option(s) Correct Type

7. Let $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s)

(a) 1056 (b) 1088 (c) 1120 (d) 1332

8. The largest value of the positive integer m for which $n^m + 1$ divides $1 + n + n^2 + ... + n^{127}$ is divisible by

(a) 8 (b) 16

(c) 32 (d) 64

9. a and c are two distinct positive real numbers such that a, b, c are in G.P. If $b - c$, $c - a$, $a - b$ are in H.P., then $a + 4b + c$ is

(a) a constant (b) independent of a

(c) independent of b (d) independent of c

10. If the first and the $(2n-1)^{\text{th}}$ term of an A.P., a G.P. and an H.P. are equal and their n^{th} terms are a, b and c respectively, then

(a) $a = b = c$ (b) $a \ge b \ge c$

(c) $a + c = b$ (d) $ac - b^2 = 0$.

11. If $\log_2(5\cdot 2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in A.P., then x is equal to

(a) $\frac{\log 5}{\log 2}$ (b) $\log_2(0.4)$

(c) $1-\frac{\log 5}{\log 2}$ (d) $\frac{\log 2}{\log 5}$

12. If a, b, c are in H.P., then the value of

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right) \text{ is}$$

(a) $\frac{2}{bc}-\frac{1}{b^2}$ (b) $\frac{1}{4}\left(\frac{3}{c^2}+\frac{2}{ca}-\frac{1}{a^2}\right)$

(c) $\frac{3}{b^2}-\frac{2}{ab}$ (d) none of these

13. For $0<\phi<\frac{\pi}{2}$, if $x=\sum_{n=0}^{\infty}\cos^{2n}\phi$, $y=\sum_{n=0}^{\infty}\sin^{2n}\phi$

and $z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$, then

(a) $xyz = xz + y$ (b) $xyz = xy + z$

(c) $xyz = x + y + z$ (d) $xyz = yz + x$

Comprehension Type

Let V_r denote the sum of first r terms of an A.P. whose first term is r and the common difference is $(2r - 1)$. Let $T_r = V_{r+1} - V_r$ for $r = 1, 2, ...$

14. The sum $V_1 + V_2 + ... + V_n$ is

(a) $\frac{1}{12}n(n+1)(3n^2-n+1)$

(b) $\frac{1}{12}n(n+1)(3n^2+n+2)$

(c) $\frac{1}{2}n(2n^2-n+1)$

(d) $\frac{1}{3}(2n^3-2n+3)$

15. T_r is always

(a) an odd number (b) an even number

(c) a prime number (d) a composite number

Matrix Match Type

16. It is given that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+....=\log 2$

and $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+....=\frac{\pi}{4}$.

Then

	Column I		Column II
A.	$\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+...$	1.	$\pi/6$
B.	$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+...$	2.	$\pi/8$
C.	$\sum_{r=1}^{\infty}\frac{1}{(2r-1)r}$	3.	log2
		4.	log4

	A	B	C
(a)	1	4	3
(b)	2	3	4
(c)	1	2	4
(d)	2	1	3

Integer Answer Type

17. If x, y, z are positive and $x + y + z = 1$, then

$\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right)$ is greater than or equal to

18. If the function f satisfies the relation $f(x + y) = f(x)\cdot f(y)$ for all natural numbers x, y, $f(1) = 2$ and $\sum_{r=1}^{n} f(a+r) = 16(2^n - 1)$, then the natural number a, is

19. The unit digit of the number of common terms to the sequence 17, 21, 25, ..., 417 and 16, 21, 26,....., 466 is

20. If one G.M. is g and two A.M.s are p and q, are inserted between two numbers a and b, then $\frac{(2p-q)(-p+2q)}{g^2}$ is equal to

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CONCEPT BOOSTERS

Class XII

Differential Equations

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

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Definition : An equation involving independent variable x, dependent variable y and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots$ is called differential equation.

- **Order of a differential equation :** The order of a differential equation is the order of the highest derivative occurring in the differential equation.
- **Degree of a differential equation :** The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.

FORMATION OF DIFFERENTIAL EQUATION

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. The equation so obtained is the differential equation of order n for the family of given curves.

(i) Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constants.

(ii) Obtain the number of arbitrary constants let there be n arbitrary constants.

(iii) Differentiate the given relation in n times with respect to x.

(iv) Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step (iii) and an equation in step (i). The equation so obtained is the desired differential equation.

VARIABLE SEPARABLE FORM

If the differential equation of the form

$$f_1(x)\,dx = f_2(y)\,dy \qquad \ldots\text{(i)}$$

where f_1 and f_2 being functions of x and y only. Thus, integrating both sides of (i), we get its solution as

$$\int f_1(x)dx = \int f_2(y)dy + c,$$

where c is an arbitrary constant.

- **Equations reducible to variable separable form:**

(i) Differential equations of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to variable separable form by the substitution $ax + by + c = Z$.

$$\therefore\quad a + b\frac{dy}{dx} = \frac{dZ}{dx}$$

$$\therefore\quad \left(\frac{dZ}{dx} - a\right)\frac{1}{b} = f(Z) \;\Rightarrow\; \frac{dZ}{dx} = a + bf(Z)$$

This is variable separable form.

(ii) Differential equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C} \text{ where } \frac{a}{A} = \frac{b}{B} = K \text{ (say)}$$

$$\therefore\quad \frac{dy}{dx} = \frac{K(Ax+By)+c}{Ax+By+C}$$

Put $Ax + By = Z \;\Rightarrow\; A + B\frac{dy}{dx} = \frac{dZ}{dx}$

$$\therefore\quad \left[\frac{dZ}{dx} - A\right]\frac{1}{B} = \frac{KZ+c}{Z+C} \;\Rightarrow\; \frac{dZ}{dx} = A + B\frac{KZ+c}{Z+C}$$

This is variable separable form and can be solved.

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HOMOGENEOUS DIFFERENTIAL EQUATION

If a first order, first degree differential equation is expressible in the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation. Such type of equations can be reduced to variable separable form by the substitution $y = vx$.

- **Equation reducible to homogeneous form :** A first order, first degree differential equation of the form $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$,
This is non-homogeneous differential equation. It can be reduced to homogeneous form by certain substitutions. Put $x = X + h$, $y = Y + k$, where h and k are constants, which are to be determined.

LINEAR DIFFERENTIAL EQUATION

- **Linear and non-linear differential equations:** A differential equation is a linear differential equation if it is expressible in the form
$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_{n-1}\frac{dy}{dx}+P_ny=Q$$
where P_0, P_1, P_2, ... , P_{n-1}, P_n and Q are either constants or functions of independent variable x. Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non linear differential equation.
- **Linear differential equation of first order :** The general form of a linear differential equation of first order is
$$\frac{dy}{dx}+Py=Q \qquad ...(i)$$
Where P and Q are functions of x (or constants)
Multiplying both sides by $e^{\int Pdx}$, we get
$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=Qe^{\int Pdx}$$
$$\Rightarrow \frac{d}{dx}\left\{ye^{\int Pdx}\right\}=Qe^{\int Pdx}$$
On integrating both sides w.r.t. x, we get
$$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c \qquad ...(ii)$$
which is the required solution, where c is the constant of integration. $e^{\int Pdx}$ is called the integrating factor. The solution (ii) in short may also be written as $y.(I.F.)=\int Q.(I.F.)\,dx+c$
- **Linear differential equation of the form $\frac{dx}{dy}+Rx=S$:** Sometimes a linear differential equation can be put in the form $\frac{dx}{dy}+Rx=S$ where R and S are functions of y (or constants). Note that y is independent variable and x is a dependent variable.
- **Equations reducible to linear form (Bernoulli's differential equation) :** The differential equation of type $\frac{dy}{dx}+Py=Qy^n$ ...(i)
where P and Q are functions of x alone or constants and n is a constant other than zero or unity, can be reduced to the linear form by dividing by y^n and then putting $y^{-n+1} = v$, as explained below.
Dividing both sides of (i) by y^n, we get
$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q$$
Putting $y^{-n+1} = v$ so that $(-n+1)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$, we get
$$\frac{1}{-n+1}\frac{dv}{dx}+Pv=Q\Rightarrow\frac{dv}{dx}+(1-n)Pv=(1-n)Q$$
which is a linear differential equation.
- **Differential equation of the form**
$$\boldsymbol{\frac{dy}{dx}+P\phi(y)=Q\psi(y)}$$
where P and Q are functions of x alone or constants.
Dividing by $\psi(y)$, we get $\frac{1}{\psi(y)}\frac{dy}{dx}+\frac{\phi(y)}{\psi(y)}P=Q$
Now put $\frac{\phi(y)}{\psi(y)}=v$, so that $\frac{d}{dx}\left\{\frac{\phi(y)}{\psi(y)}\right\}=\frac{dv}{dx}$
or $\frac{dv}{dx}=k\cdot\frac{1}{\psi(y)}\frac{dy}{dx}$, where k is constant
We get $\frac{dv}{dx}+kPv=kQ$
which is linear differential equation.

APPLICATION OF DIFFERENTIAL EQUATION

Differential equation is applied in various practical fields of life. It is used to define various physical laws and quantities. It is widely used in physics, chemistry, engineering etc.

Some important fields of application are

- Rate of change
- Geometrical problems etc.

MISCELLANEOUS DIFFERENTIAL EQUATION

A special type of second order differential equation :

$$\frac{d^2y}{dx^2} = f(x) \quad \text{...(i)}$$

Equation (i) may be re-written as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$

$\Rightarrow \quad d\left(\frac{dy}{dx}\right) = f(x)dx$

Integrating, $\frac{dy}{dx} = \int f(x)dx + c_1$

i.e. $\frac{dy}{dx} = F(x) + c_1$ ...(ii)

Where $F(x) = \int f(x)dx$

From (ii), $dy = F(x)dx + c_1dx$

Integrating, $y = \int F(x)dx + c_1x + c_2$

$\therefore \quad y = H(x) + c_1x + c_2$

where $H(x) = \int F(x)dx$ and c_1 and c_2 are arbitrary constants.

PROBLEMS

Single Correct Answer Type

1. The degree and order of the differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = x^2$ respectively is

(a) 3, 2 (b) 1, 1 (c) 4, 3 (d) 4, 4

2. If m and n are the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\left(\frac{d^3y}{dx^3}\right)} + \frac{d^3y}{dx^3} = x^2 - 1$, then (m, n) is

(a) (3, 5) (b) (3, 1) (c) (3, 3) (d) (3, 2)

3. The order and degree of the differential equation $\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$ are respectively

(a) 2, 2 (b) 2, 3

(c) 2, 1 (d) none of these

4. The differential equation for all the straight lines which are at a unit distance from the origin is

(a) $\left(y - x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$

(b) $\left(y + x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(c) $\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(d) $\left(y + x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$

5. The differential equation whose solution is $y = c_1\cos ax + c_2\sin ax$ is (where c_1, c_2 are arbitrary constants)

(a) $\frac{d^2y}{dx^2} + y^2 = 0$ (b) $\frac{d^2y}{dx^2} + a^2y = 0$

(c) $\frac{d^2y}{dx^2} + ay^2 = 0$ (d) $\frac{d^2y}{dx^2} - a^2y = 0$

6. The differential equation of the family of curves $y^2 = 4a(x + a)$, where a is an arbitrary constant, is

(a) $y\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$

(b) $y\left[1 - \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ (d) $\left(\frac{dy}{dx}\right)^3 + 3\frac{dy}{dx} + y = 0$

7. The differential equation of all parabolas whose axes are parallel to y-axis is

(a) $\frac{d^3y}{dx^3} = 0$ (b) $\frac{d^2x}{dy^2} = c$

(c) $\frac{d^3y}{dx^3} + \frac{d^2x}{dy^2} = 0$ (d) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = c$

8. Family of curves $y = e^x(A\cos x + B\sin x)$, represents the differential equation

(a) $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - y$ (b) $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2y$

(c) $\frac{d^2y}{dx^2} = \frac{dy}{dx} - 2y$ (d) $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + y$

9. $y = ae^{mx} + be^{-mx}$ satisfies which of the following differential equation

(a) $\frac{dy}{dx} - my = 0$ (b) $\frac{dy}{dx} + my = 0$

(c) $\frac{d^2y}{dx^2}+m^2y=0$ (d) $\frac{d^2y}{dx^2}-m^2y=0$

10. Differential equation of $y = \sec(\tan^{-1}x)$ is

(a) $(1+x^2)\frac{dy}{dx}=y+x$ (b) $(1+x^2)\frac{dy}{dx}=y-x$

(c) $(1+x^2)\frac{dy}{dx}=xy$ (d) $(1+x^2)\frac{dy}{dx}=\frac{x}{y}$

11. If $x = \sin t$, $y = \cos pt$, then

(a) $(1 - x^2)y_2 + xy_1 + p^2y = 0$
(b) $(1 - x^2)y_2 + xy_1 - p^2y = 0$
(c) $(1 + x^2)y_2 - xy_1 + p^2y = 0$
(d) $(1 - x^2)y_2 - xy_1 + p^2y = 0$

12. The solution of the differential equation $3e^x\tan y dx + (1 - e^x)\sec^2 y dy = 0$ is

(a) $\tan y = c(1 - e^x)^3$ (b) $(1 - e^x)^3 \tan y = c$
(c) $\tan y = c(1 - e^x)$ (d) $(1 - e^x) \tan y = c$

13. The solution of the equation $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$ is

(a) $e^y=e^x+\frac{x^3}{3}+c$ (b) $e^y=e^x+2x+c$

(c) $e^y=e^x+x^3+c$ (d) $y = e^x + c$

14. The solution of the differential equation $\frac{dy}{dx}=e^x+\cos x+x+\tan x$ is

(a) $y=e^x+\sin x+\frac{x^2}{2}+\log\cos x+c$

(b) $y=e^x+\sin x+\frac{x^2}{2}+\log\sec x+c$

(c) $y=e^x-\sin x+\frac{x^2}{2}+\log\cos x+c$

(d) $y=e^x-\sin x+\frac{x^2}{2}+\log\sec x+c$

15. The solution of the differential equation $(\sin x + \cos x)dy + (\cos x - \sin x)dx = 0$ is

(a) $e^x(\sin x + \cos x) + c = 0$
(b) $e^y(\sin x + \cos x) = c$
(c) $e^y(\cos x - \sin x) = c$
(d) $e^x(\sin x - \cos x) = c$

16. The solution of the differential equation $\frac{dy}{dx}=1+x+y+xy$ is

(a) $\log(1+y)=x+\frac{x^2}{2}+c$ (b) $(1+y)^2=x+\frac{x^2}{2}+c$

(c) $\log(1 + y) = \log(1 + x) + c$
(d) none of these

17. The solution of the differential equation $(x^2-yx^2)\frac{dy}{dx}+y^2+xy^2=0$ is

(a) $\log\left(\frac{x}{y}\right)=\frac{1}{x}+\frac{1}{y}+c$ (b) $\log\left(\frac{y}{x}\right)=\frac{1}{x}+\frac{1}{y}+c$

(c) $\log(xy)=\frac{1}{x}+\frac{1}{y}+c$ (d) $\log(xy)+\frac{1}{x}+\frac{1}{y}=c$

18. The solution of the differential equation $x\sec y\frac{dy}{dx}=1$ is

(a) $x\sec y \tan y = c$ (b) $cx = \sec y + \tan y$
(c) $cy = \sec x \tan x$ (d) $cy = \sec x + \tan x$

19. The solution of differential equation $x\frac{dy}{dx}+y=y^2$ is

(a) $y = 1 + cxy$ (b) $y = \log\{cxy\}$
(c) $y + 1 = cxy$ (d) $y = c + xy$

20. If $\frac{dy}{dx}=\frac{xy+y}{xy+x}$, then the solution of the differential equation is

(a) $y = xe^x + c$ (b) $y = e^x + c$
(c) $y = Axe^{x-y}$ (d) $y = x + A$

21. The general solution of the equation $(e^y + 1)\cos x dx + e^y \sin x dy = 0$ is

(a) $(e^y + 1)\cos x = c$ (b) $(e^y - 1)\sin x = c$
(c) $(e^y + 1)\sin x = c$ (d) none of these

22. The solution of the differential equation $x^2dy = -2xydx$ is

(a) $xy^2 = c$ (b) $x^2y^2 = c$ (c) $x^2y = c$ (d) $xy = c$

23. The solution of the equation $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$ is

(a) $\tan(x + y) + \sec(x + y) = x + c$
(b) $\tan(x + y) - \sec(x + y) = x + c$
(c) $\tan(x + y) + \sec(x + y) + x + c = 0$
(d) none of these

24. The solution of the differential equation $\frac{dy}{dx}=\frac{x-y+3}{2(x-y)+5}$ is

(a) $2(x - y) + \log(x - y) = x + c$
(b) $2(x - y) - \log(x - y + 2) = x + c$
(c) $2(x - y) + \log(x - y + 2) = x + c$
(d) none of these

25. The solution of the differential equation $(x - y^2x)dx = (y - x^2y)dy$ is

(a) $(1 - y^2) = c^2(1 - x^2)$
(b) $(1 + y^2) = c^2(1 - x^2)$
(c) $(1 + y^2) = c^2(1 + x^2)$
(d) none of these

26. The solution of $(\operatorname{cosec}x \log y)dy + (x^2y)dx = 0$ is

(a) $\frac{\log y}{2} + (2 - x^2)\cos x + 2\sin x = c$

(b) $\left(\frac{\log y}{2}\right)^2 + (2 - x^2)\cos x + 2x\sin x = c$

(c) $\frac{(\log y)^2}{2} + (2 - x^2)\cos x + 2x\sin x = c$

(d) none of these

27. The solution of the differential equation

$$xy\frac{dy}{dx} = \frac{(1+y^2)(1+x+x^2)}{(1+x^2)}$$ is

(a) $\frac{1}{2}\log(1+y^2) = \log x - \tan^{-1}x + c$

(b) $\frac{1}{2}\log(1+y^2) = \log x + \tan^{-1}x + c$

(c) $\log(1 + y^2) = \log x - \tan^{-1}x + c$
(d) $\log(1 + y^2) = \log x + \tan^{-1}x + c$

28. The solution of $(x\sqrt{1+y^2})dx + (y\sqrt{1+x^2})dy = 0$ is

(a) $\sqrt{1+x^2} + \sqrt{1+y^2} = c$

(b) $\sqrt{1+x^2} - \sqrt{1+y^2} = c$

(c) $(1+x^2)^{3/2} + (1+y^2)^{3/2} = c$

(d) none of these

29. The solution of the differential equation $(1 + x^2)(1 + y)dy + (1 + x)(1 + y^2)dx = 0$ is

(a) $\tan^{-1}x + \log(1 + x^2) + \tan^{-1}y + \log(1 + y^2) = c$

(b) $\tan^{-1}x - \frac{1}{2}\log(1+x^2) + \tan^{-1}y - \frac{1}{2}\log(1+y^2) = c$

(c) $\tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = c$

(d) none of these

30. The solution of the differential equation $\cos y \log(\sec x + \tan x)dx = \cos x \log(\sec y + \tan y)dy$ is

(a) $\sec^2x + \sec^2y = c$ (b) $\sec x + \sec y = c$
(c) $\sec x - \sec y = c$ (d) none of these

31. The solution of the equation $\frac{dy}{dx} = \frac{y^2 - y - 2}{x^2 + 2x - 3}$ is

(a) $\frac{1}{3}\log\left|\frac{y-2}{y+1}\right| = \frac{1}{4}\log\left|\frac{x+3}{x-1}\right| + c$

(b) $\frac{1}{3}\log\left|\frac{y+1}{y-2}\right| = \frac{1}{4}\log\left|\frac{x-1}{x+3}\right| + c$

(c) $4\log\left|\frac{y-2}{y+1}\right| = 3\log\left|\frac{x-1}{x+3}\right| + c$

(d) none of these

32. Solution of $ydx - xdy = x^2ydx$ is

(a) $ye^{x^2} = cx^2$ (b) $ye^{-x^2} = cx^2$
(c) $y^2e^{x^2} = cx^2$ (d) $y^2e^{-x^2} = cx^2$

33. The solution of $\frac{dy}{dx} = 2^{y-x}$ is

(a) $2^x + 2^y = c$ (b) $2^x - 2^y = c$
(c) $\frac{1}{2^x} - \frac{1}{2^y} = c$ (d) $x + y = c$

Multiple Correct Answer Type

34. The curve whose subtangent is 'n' times the abscissa of the point of contact and passes through the point (2, 3), then

(a) for $n = 1$ equation of the curve is $2y = 3x$
(b) for $n = 1$ equation of the curve is $2y^2 = 9x$
(c) for $n = 2$ equation of the curve is $2y = 3x$
(d) for $n = 2$ equation of the curve is $2y^2 = 9x$

35. If the solution of the equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0$ given that for $t = 0$, $x = 0$ and $\frac{dx}{dt} = 12$ is in the form $x = Ae^{-3t} + Be^{-t}$ then

(a) $A + B = 0$ (b) $A + B = 12$
(c) $|AB| = 36$ (d) $|AB| = 49$

36. If differential equation is formed to the family of all the central conics centred at origin, then

(a) order = 2 (b) order = 3
(c) degree = 1 (d) degree = order = 3

37. Solution of the differential equation :

$$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\sin^2(x^2+y^2)}{y^3}$$ is

(a) $-\cot(x^2 + y^2) = \left(\frac{x}{y}\right)^2 + c$

(b) $\frac{y^2}{x^2 + y^2c} = -\tan(x^2 + y^2)$

(c) $-\cot(x^2+y^2)=\left(\frac{y}{x}\right)^2+c$

(d) $\frac{y^2+x^2c}{x^2}=-\tan^2(x^2+y^2)$

38. Let C be a curve such that the normal at any point P on it meets x-axis and y-axis at A and B respectively. If $BP : PA = 1 : 2$ (internally) and the curve passes through the point (0, 4) then which of the following alternative(s) is/are correct?

(a) The curves passes through $(\sqrt{10},-6)$

(b) The equation of tangent at $(4,4\sqrt{3})$ is $2x+\sqrt{3}y=20$

(c) The differential equation for the curve is $yy'+2x=0$

(d) The curve represent a hyperbola

39. If solution of $x^2\frac{d^2y}{dx^2}-9x\frac{dy}{dx}+21y=0$ is of the form $y = c_1x^m + c_2x^n$ (c_1, c_2 are arbitrary constants, $m, n \in N$) then values of m, n can be

(a) $m = 3, n = 1$ (b) $m = 3, n = 7$

(c) $m = 7, n = 3$ (d) $m = 1, n = 7$

40. If the solution of $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=x,\ y(0)=y(1)=1$ is given by $y^2 = f(x)$ then

(a) $f(x)$ is monotonically increasing $\forall\, x \in (1, \infty)$

(b) $f(x) = 0$ has only one real root

(c) $f(x)$ is neither even nor odd

(d) $f(x)$ has 3 real roots

41. Solution of the differential equation

$(3\tan x + 4\cot y - 7)\sin^2 y\, dx - (4\tan x + 7\cot y - 5)\cos^2 x dy = 0$ is

(a) $\frac{3}{2}\cot^2 x-7\cot x+\frac{7}{2}\tan^2 y-5\tan y+4\cot x\cdot\tan y=c$

(b) $\frac{3}{2}\tan^2 x-7\tan x+\frac{7}{2}\cot^2 y-5\cot y+4\tan x\cdot\cot y=c$

(c) $3\tan^2 y - 14\cot x\cdot\tan^2 y + 7\cot^2 x - 10\tan y\cot^2 x + 8\cot x\cdot\tan y + 2c\cot^2 x\tan^2 y = 0.$

(d) $3\cot^2 y - 14\cot x\cdot\cot^2 y + 7\cot^2 x + 10\cot y\tan^2 x + 8\tan x\cdot\cot y = 0.$

Comprehension Type

Paragraph for Q. No. 42 to 44

Consider a tank which initially holds V_0 ltr. of brine that contains a lb of salt. Another brine solution, containing b lb of salt/ltr., is poured into the tank at the rate of e ltr./min while, simultaneously, the well-stirred solution leaves the tank at the rate of f ltr./min. The problem is to find the amount of salt in the tank at any time t.

Let Q denote the amount of salt in the tank at any time. The time rate of change of Q, dQ/dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of be lb/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added et minus the volume of brine removed ft. Thus, the volume of brine at any time is

$$V_0 + et - ft \quad \text{... (a)}$$

The concentration of salt in the tank at any time is $Q(V_0 + et - ft)$, from which it follows that salt leaves the tank at the rate of $f\left(\frac{Q}{V_0+et-ft}\right)$ lb/min

Thus, $$\frac{dQ}{dt}=be-f\left(\frac{Q}{V_0+et-ft}\right) \quad \text{... (b)}$$

or $$\frac{dQ}{dt}+\frac{f}{V_0+et-ft}Q=be$$

42. A tank initially holds 100 ltr. of a brine solution containing 20 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 5 ltr./min, while the well-stirred mixture leaves the tank at the same rate. Then the amount of salt in the tank after 20 min.

(a) $20/e$ (b) $10/e$ (c) $40/e^2$ (d) $5/e$

43. A 50 ltr. tank initially contains 10 ltr. of fresh water. At $t = 0$, a brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 ltr./min, while the well-stirred mixture leaves the tank at the rate of 2 ltr./min. Then the amount of time required for overflow to occur is

(a) 30 min (b) 20 min (c) 10 min (d) 40 min

44. In the above question, the amount of salt in the tank at the moment of overflow is

(a) 20 lb (b) 50 lb

(c) 30 lb (d) None of these

Paragraph for Q. No. 45 to 47

Newton's law of cooling states that the rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K.

We can write it as $\frac{dT}{dt}=-k(T-290), k>0,$ a constant.

where T is temperature of substance.

45. The substance cools from 370 K to 330 K in 10 min, then

(a) $T = 290 + 160e^{-kt}$ (b) $T = 290 + 80e^{-kt}$
(c) $T = 290 + 40e^{-kt}$ (d) $T = 290 + 20e^{-kt}$

46. In the above question, the value of k must be

(a) ln2 (b) $\frac{\ln 2}{40}$ (c) $\frac{\ln 2}{20}$ (d) $\frac{\ln 2}{10}$

47. The time taken when temperature be 295 K is

(a) 10 min (b) 20 min
(c) 40 min (d) 80 min

Matrix – Match Type

48. Match the following :

	Column-I		Column-II
A.	If order and degree of the differential equation formed by differentiating and eliminating the constants from $y = a \sin^2 x + b \cos^2 x + c \sin 2x + d \cos 2x$, where a, b, c, d are arbitrary constants are represented by O and D, then	P.	$O + 2D = 5$
B.	The order and degree of the differential equation, whose general solution is given by $y = (c_1 + c_2)\sin(x + c_3) - c_4 e^{x+c_5+c_6}$, where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants, are O and D, then	Q.	$2^O + 3^D = 5$
C.	The order and degree of the differential equation satisfying $\sqrt{(1+x^2)} + \sqrt{(1+y^2)} = A(x\sqrt{(1+y^2)} + y\sqrt{(1+x^2)})$ are O and D, then	R.	$O = D$
		S.	$O^D + D^O = 4$

49. Match the following :

	Column-I		Column-II
A.	If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\dfrac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = K$, then the value of $K/3$ is	P.	3
B.	Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is	Q.	4
C.	If real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4 y\frac{dy}{dx} + y^4 - 4x^6$ into a homogeneous equation, then the value of $2m$ is	R.	2

Integer Answer Type

50. If the solution of the differential equation $e^{3x}(p - 1) + p^3 e^{2y} = 0$ is in the form $e^y = ce^x + c^k$ then $k =$ __ (where $p = \frac{dy}{dx}$ and 'c' is arbitrary constant)

51. If the solution of the differential equation $y^2(y - xp) = x^4p^2$ is in the form $\frac{1}{y} = \frac{c}{x} + c^k$ then $k =$ __ (where $p = \frac{dy}{dx}$ and 'c' is arbitrary constant)

52. If the solution of the differential equation $y = 2px + y^2p^3$ is in the form $y^2 = 2cx + c^k$ then $k =$ __(where $p = \frac{dy}{dx}$ and 'c' is arbitrary constant)

53. The solution of $x^3\frac{dy}{dx} + 4x^2 \tan y = e^x \sec y$ satisfying $y(1) = 0$ is $\sin y = e^x(x - 1)x^{-k}$ then $k =$

54. Differential equation, satisfying $y = (\sin^{-1}x)^2 + A(\cos^{-1}x) + B$, where A and B are arbitrary constants is

$$(p - x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} = q \text{ then } p + q = \_\_$$

55. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from normal at any point 'P' of the curve is equal to the distance of P from the x-axis is a circle with radius ____.

SOLUTIONS

1. (a) : Given differential equation,

$$x\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = x^2$$

Hence, order = 2 and degree = 3

2. (d) : The order (m) of the given equation is 3 and degree (n) of the given equation is 2. Therefore $m = 3$ and $n = 2$.

3. (a) : $\rho.\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

$$\Rightarrow \left(\rho.\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$\therefore$ Order = 2, degree = 2.

4. (c) : Since the equation of lines whose distance from origin is unit, is given by

$x\cos\alpha + y\sin\alpha = 1$ ...(i)

Differentiate w.r.t. x, we get

$$\cos\alpha+\frac{dy}{dx}\sin\alpha=0 \qquad ...(ii)$$

On solving (i) and (ii),

$$\sin\alpha\left(y-x\frac{dy}{dx}\right)=1 \Rightarrow \left(y-x\frac{dy}{dx}\right)=\mathrm{cosec}\alpha \qquad ...(iii)$$

$$\text{Also (ii)} \Rightarrow \frac{dy}{dx}=-\cot\alpha \Rightarrow \left(\frac{dy}{dx}\right)^2=\cot^2\alpha \qquad ...(iv)$$

Therefore by (iii) and (iv), $1+\left(\frac{dy}{dx}\right)^2=\left(y-x\frac{dy}{dx}\right)^2$.

5. (b) : Since, $y = c_1\cos ax + c_2\sin ax$

Differentiate w.r.t. x, we get

$$\frac{dy}{dx}=-c_1a\sin ax+c_2a\cos ax$$

$$\Rightarrow \frac{d^2y}{dx^2}=-c_1a^2\cos ax-c_2a^2\sin ax$$

$$\frac{d^2y}{dx^2}=-a^2(c_1\cos ax+c_2\sin ax)\Rightarrow\frac{d^2y}{dx^2}=-a^2y$$

or $\frac{d^2y}{dx^2}+a^2y=0.$

6. (b) : Given $y^2 = 4a(x + a)$ ...(i)

differentiating, we get

$$2y\left(\frac{dy}{dx}\right)=4a \qquad ...(ii)$$

Eliminating a from (i) and (ii),

Required differentially equation is

$$y\left[1-\left(\frac{dy}{dx}\right)^2\right]=2x\frac{dy}{dx}.$$

7. (a) : The equation of a member of the family of parabolas having axis parallel to y-axis is

$y = Ax^2 + Bx + C$ ...(i)

where A, B, C are arbitrary constants.

Differentiating (i) w.r.t. x, we get $\frac{dy}{dx}=2Ax+B$ ...(ii)

Differentiating (ii) w.r.t. x gives

$$\frac{d^2y}{dx^2}=2A \qquad ...(iii)$$

Differentiating (iii) w.r.t. x again, we get $\frac{d^3y}{dx^3}=0.$

8. (b) : Given, $y = e^xA\cos x + e^xB\sin x$

$$\Rightarrow \frac{dy}{dx}=Ae^x\cos x-Ae^x\sin x+Be^x\sin x+Be^x\cos x$$

$$\frac{dy}{dx}=(A+B)e^x\cos x+(B-A)e^x\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2}=(A+B)e^x\cos x-e^x(A+B)\sin x + (B-A)e^x\sin x + (B-A)e^x\cos x$$

$$\frac{d^2y}{dx^2}=2Be^x\cos x-2Ae^x\sin x.$$

Hence, $\frac{d^2y}{dx^2}=2\frac{dy}{dx}-2y.$

9. (d) : $y = ae^{mx} + be^{-mx}$

$$\Rightarrow \frac{dy}{dx}=mae^{mx}-mbe^{-mx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=m^2ae^{mx}+m^2be^{-mx}=m^2y$$

or $\frac{d^2y}{dx^2}-m^2y=0.$

10. (c) : $y = \sec(\tan^{-1}x)$

$$\frac{dy}{dx}=\sec(\tan^{-1}x)\tan(\tan^{-1}x).\frac{1}{1+x^2}=\frac{xy}{1+x^2}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}=xy.$$

11. (d) : $x = \sin t$, $y = \cos pt$

$$\frac{dx}{dt}=\cos t;\frac{dy}{dt}=-p\sin pt\Rightarrow\frac{dy}{dx}=\frac{-p\sin pt}{\cos t}$$

$$\frac{d^2y}{dx^2}=\frac{-\cos t\,p^2\cos pt(dt/dx)-p\sin pt\sin t(dt/dx)}{\cos^2 t}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$$

or $(1 - x^2)y_2 - xy_1 + p^2y = 0.$

12. (a) : Given differential equation can be written in the form of

$$\frac{\sec^2y}{\tan y}dy=-3\frac{e^x}{1-e^x}dx$$

On integrating,

$$\int \frac{\sec^2 y}{\tan y} dy = -3\int \frac{e^x}{1-e^x} dx$$

$\Rightarrow$ $\log(\tan y) = 3\log(1 - e^x) + \log c$

$\Rightarrow$ $\tan y = c(1 - e^x)^3$.

13. (a) : $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y} = e^{-y}(e^x + x^2)$

$\Rightarrow$ $e^y dy = (x^2 + e^x)dx$

On integrating both sides, we get

$$e^y = \frac{x^3}{3} + e^x + c.$$

14. (b) : $\frac{dy}{dx} = e^x + \cos x + x + \tan x$

On integrating both sides, we get

$$y = e^x + \sin x + \frac{x^2}{2} + \log \sec x + c.$$

15. (b) : $\frac{dy}{dx} = -\frac{\cos x - \sin x}{\sin x + \cos x}$

$\Rightarrow dy = -\left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)dx$

On integrating both sides, we get

$\Rightarrow$ $y = -\log(\sin x + \cos x) + \log c$

$\Rightarrow y = \log\left(\frac{c}{\sin x + \cos x}\right) \Rightarrow e^y(\sin x + \cos x) = c.$

16. (a) : $\frac{dy}{dx} = 1 + x + y + xy$

$\Rightarrow \frac{dy}{dx} = (1+x)(1+y) \Rightarrow \frac{dy}{1+y} = (1+x)dx$

On integrating, we get $\log(1+y) = \frac{x^2}{2} + x + c.$

17. (a) : The given equation is

$(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0 \Rightarrow \frac{1-y}{y^2}dy + \frac{1+x}{x^2}dx = 0$

$\Rightarrow \left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \left(\frac{1}{x^2} + \frac{1}{x}\right)dx = 0$

On integrating, we get the required solution

$$\log\left(\frac{x}{y}\right) = \frac{1}{x} + \frac{1}{y} + c.$$

18. (b) : We have, $x \sec y \frac{dy}{dx} = 1 \Rightarrow \sec y\, dy = \frac{dx}{x}$

On integrating both sides, we get

$\log(\sec y + \tan y) = \log x + \log c \Rightarrow \sec y + \tan y = cx.$

19. (a) : We have, $x\frac{dy}{dx} + y = y^2 \Rightarrow x\frac{dy}{dx} = y^2 - y$

$\Rightarrow \frac{dy}{y^2 - y} = \frac{dx}{x} \Rightarrow \left[\frac{1}{y-1} - \frac{1}{y}\right]dy = \frac{dx}{x}$

On integrating, we get, $\log(y - 1) - \log y = \log x + \log c$

$\Rightarrow \frac{y-1}{y} = xc \Rightarrow y = 1 + cxy.$

20. (c) : $\frac{dy}{dx} = \frac{xy + y}{xy + x} \Rightarrow \left(\frac{1+y}{y}\right)dy = \left(\frac{1+x}{x}\right)dx$

On integrating both sides, we get

$\log y + y = \log x + x + \log A$

$\Rightarrow \log\left(\frac{y}{Ax}\right) = x - y \Rightarrow y = Axe^{x-y}$

21. (c) : $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$\Rightarrow \frac{e^y dy}{e^y + 1} + \frac{\cos x}{\sin x}dx = 0$

On integrating both sides, we get

$\log(e^y + 1) + \log(\sin x) = \log c \Rightarrow (e^y + 1)\sin x = c$

22. (c) : We have, $x^2 dy = -2xy dx \Rightarrow \frac{1}{y}dy = -\frac{2x}{x^2}dx$

On integrating, $\log y = -2\log x + \log c$

$\Rightarrow$ $\log y = \log x^{-2} + \log c \Rightarrow \log yx^2 = \log c$ or $yx^2 = c$.

23. (b) : Here, $\frac{dy}{dx} = \sin(x+y)$ ...(i)

Put $x + y = v$ $\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\therefore$ (i) becomes $\frac{dv}{1 + \sin v} = dx$

Now on integrating both sides, we get

$\tan v - \sec v = x + c$ or $\tan(x + y) - \sec(x + y) = x + c$.

24. (c) : Here, $\frac{dy}{dx} = \frac{x - y + 3}{2(x-y) + 5}$

Let $x - y = v$ and $\frac{dy}{dx} = 1 - \frac{dv}{dx}$

Thus the differential equation reduces to $\frac{dv}{dx} = \frac{v+2}{2v+5}$

$\Rightarrow \int \frac{2v+5}{v+2}dv = \int dx \Rightarrow \int \left[2 + \frac{1}{(v+2)}\right]dv = \int dx$

$\Rightarrow 2v + \log(v+2) = x + c$

or $2(x - y) + \log(x - y + 2) = x + c$.

25. (a) : Given equation can be written as

$$\frac{x}{1-x^2}dx = \frac{y}{1-y^2}dy$$

On integrating we get

$$-\frac{1}{2}\log(1-x^2) = -\frac{1}{2}\log(1-y^2) + \log c$$

$$\Rightarrow \log(1-x^2)-\log(1-y^2)=-2\log c \Rightarrow \frac{1-x^2}{1-y^2}=c^{-2}$$

Hence, $(1-y^2)=c^2(1-x^2)$.

26. (c) : $(\operatorname{cosec}x \log y)dy+(x^2y)dx=0$

$$\Rightarrow \frac{1}{y}\log y\,dy=-x^2\sin x\,dx$$

On integrating both sides, we get

$$\frac{(\log y)^2}{2}+[x^2(-\cos x)+\int 2x\cos x\,dx]=c$$

$$\Rightarrow \frac{(\log y)^2}{2}+(2-x^2)\cos x+2x\sin x=c.$$

27. (b) : $xy\frac{dy}{dx}=\frac{(1+y^2)(1+x+x^2)}{(1+x^2)}$

$$\Rightarrow \int\frac{y\,dy}{1+y^2}=\int\frac{(1+x+x^2)}{x(1+x^2)}dx=\int\frac{1}{x}dx+\int\frac{dx}{1+x^2}$$

$$\Rightarrow \frac{1}{2}\log(1+y^2)=\log x+\tan^{-1}x+c.$$

28. (a) : Given equation is,

$$(x\sqrt{1+y^2})dx+(y\sqrt{1+x^2})dy=0$$

$$\Rightarrow x\sqrt{1+y^2}\,dx=-y\sqrt{1+x^2}\,dy$$

$$\Rightarrow \int\frac{x}{\sqrt{1+x^2}}dx=-\int\frac{y}{\sqrt{1+y^2}}dy$$

$$\Rightarrow \sqrt{1+x^2}+\sqrt{1+y^2}=c.$$

29. (c) : Given equation is,

$(1+x^2)(1+y)dy+(1+x)(1+y^2)dx=0$

$$\Rightarrow \frac{(1+y)}{(1+y^2)}dy=-\frac{(1+x)}{(1+x^2)}dx$$

$$\Rightarrow \int\left[\frac{1}{1+y^2}+\frac{y}{1+y^2}\right]dy+\int\left[\frac{1}{1+x^2}+\frac{x}{1+x^2}\right]dx-c=0$$

$$\Rightarrow \tan^{-1}y+\frac{1}{2}\log(1+y^2)+\tan^{-1}x+\frac{1}{2}\log(1+x^2)=c.$$

30. (d) : $\cos y\log(\sec x+\tan x)dx=\cos x\log(\sec y+\tan y)dy$

$$\Rightarrow \int\sec y\log(\sec y+\tan y)dy=\int\sec x\log(\sec x+\tan x)dx$$

$$\Rightarrow \frac{[\log(\sec x+\tan x)]^2}{2}=\frac{[\log(\sec y+\tan y)]^2}{2}+c.$$

31. (c) : $\frac{dy}{dx}=\frac{y^2-y-2}{x^2+2x-3}$

$$\Rightarrow \frac{dy}{(y-2)(y+1)}=\frac{dx}{(x+3)(x-1)}$$

$$\Rightarrow \int\frac{dy}{(y-2)(y+1)}=\int\frac{dx}{(x+3)(x-1)}$$

$$\Rightarrow \frac{1}{3}\int\left(\frac{1}{y-2}-\frac{1}{y+1}\right)dy=\frac{1}{4}\int\left(\frac{1}{x-1}-\frac{1}{x+3}\right)dx$$

$$\Rightarrow 4\log\left|\frac{y-2}{y+1}\right|=3\log\left|\frac{x-1}{x+3}\right|+c.$$

32. (c): Given equation can be written as $\left(\frac{1-x^2}{x}\right)dx=\frac{dy}{y}$

On integration, we get $\log x-\frac{x^2}{2}=\log y+\log c_1$

$$\Rightarrow \log x^2-\log y^2+\log c=x^2 \Rightarrow \log\frac{cx^2}{y^2}=x^2$$

$$\Rightarrow \frac{cx^2}{y^2}=e^{x^2} \Rightarrow cx^2=y^2e^{x^2}.$$

33. (c) : Given, $\frac{dy}{dx}=2^{y-x}=\frac{2^y}{2^x}$ or $\frac{dy}{2^y}=\frac{dx}{2^x}$

Integrating both sides, $\int\frac{dy}{2^y}=\int\frac{dx}{2^x}$

$$\Rightarrow -2^{-y}\log 2=-2^{-x}\log 2+c_1$$

$$\Rightarrow \frac{\log 2}{2^x}-\frac{\log 2}{2^y}=c_1 \Rightarrow \frac{1}{2^x}-\frac{1}{2^y}=\frac{c_1}{\log 2}=c.$$

34. (a, d)

35. (a, c) : $x=Ae^{-3t}+Be^{-t} \Rightarrow \frac{dx}{dt}=-3Ae^{-3t}-Be^{-t}$

At $t=0$, $x=0 \Rightarrow 0=A+B$ ...(i)

At $t=0, \frac{dx}{dt}=12 \Rightarrow 12=-3A-B$ ...(ii)

Solving (i) and (ii), we get $A=-6$, $B=6$

36. (b, c) : Equation of such conics are

$ax^2+by^2+cxy=1 \Rightarrow$ Order $=3$

Degree $=1$(no parameters is being repeated)

37. (a, b) : $\frac{x\,dx+y\,dy}{y\,dx-x\,dy}=\frac{x\sin^2(x^2+y^2)}{y^3}$

$$\Rightarrow \frac{1}{2}\operatorname{cosec}^2(x^2+y^2)d(x^2+y^2)=\frac{x}{y}d\left(\frac{x}{y}\right)$$

On integrating, we get

$$\Rightarrow -\frac{1}{2}\cot(x^2+y^2)=\frac{1}{2}\left(\frac{x}{y}\right)^2+K$$

$$\Rightarrow -\cot(x^2+y^2)=\frac{x^2}{y^2}+c, \text{ where } c=2K$$

$$\Rightarrow \frac{y^2}{x^2+cy^2}=-\tan(x^2+y^2)$$

38. (a, d) : The equation of normal at $P(x, y)$ is

$$(Y-y)=\frac{-1}{\frac{dy}{dx}}(X-x)$$

$$\therefore\ A\left(x+y\frac{dy}{dx},0\right) \text{and } B\left(0, y+\frac{x}{\frac{dy}{dx}}\right)$$

$$\text{Now, } \frac{1\left(x+y\frac{dy}{dx}\right)+2(0)}{1+2}=x \Rightarrow x+y\frac{dy}{dx}=3x$$

$$\Rightarrow\ y\frac{dy}{dx}=2x \qquad \text{...(i)}$$

$$\Rightarrow \int ydy=\int 2xdx \Rightarrow \frac{y^2}{2}=x^2+C$$

Also (0,4) satisfy it, so $C = 8$

$\therefore$ $y^2 = 2x^2 + 16$ which represents a hyperbola.

$$\text{Also } \left.\frac{dy}{dx}\right]_{(4,4\sqrt{3})}=\frac{2(4)}{4\sqrt{3}}=\frac{2}{\sqrt{3}}$$

39. (b, c) : $y = c_1x^m + c_2x^n$

$\Rightarrow$ $y_1 = mc_1x^{m-1} + nc_2x^{n-1}$

$\Rightarrow$ $y_2 = m(m-1)c_1x^{m-2} + n(n-1)c_2x^{n-2}$

$\therefore$ On substituting in given equation, we get

$m^2 - 10m + 21 = 0$, $n^2 - 10n + 21 = 0$

$\Rightarrow$ $m = 3$ or 7 and $n = 3$ or 7

40. (a, b, c) : Given $\frac{d}{dx}\left(y\frac{dy}{dx}\right)=x \Rightarrow y\frac{dy}{dx}=\frac{x^2}{2}+c$

$$\Rightarrow\ \frac{d}{dx}\left(\frac{y^2}{2}\right)=\frac{x^2}{2}+c$$

$$\Rightarrow\ y^2=\frac{x^3}{3}+\alpha x+\beta \text{ where } \alpha = 2c$$

given $y(0) = 1$, $y(1) = 1$

$$\Rightarrow\ \beta=1, \alpha=\frac{-1}{3}$$

$$\therefore\ y^2=f(x)=\frac{x^3-x+3}{3} \Rightarrow f'(x)=\frac{3x^2-1}{3}>0 \text{ for } x>1$$

$\therefore$ $f(x)\uparrow$

$$f\left(\frac{1}{\sqrt{3}}\right)f\left(-\frac{1}{\sqrt{3}}\right)>0$$

$\therefore$ $f(x) = 0$ has only one real root.

41. (b, c) : Dividing throughout by $\cos^2x \sin^2y$, the given differential equation becomes

$(3 \tan x + 4 \cot y - 7)\sec^2x\, dx - (4\tan x + 7 \cot y - 5) \operatorname{cosec}^2y\, dy = 0$

$\Rightarrow$ $3\tan x \sec^2x\, dx - 7\sec^2x\, dx - 7\cot y \operatorname{cosec}^2y\, dy + 5\operatorname{cosec}^2y dy + 4\cot y \sec^2x\, dx - 4\tan x \operatorname{cosec}^2y\, dy = 0$

$\Rightarrow$ $3 \tan x\, d(\tan x) - 7d(\tan x) + 7\cot y\, d(\cot y) - 5d(\cot y) + 4d(\tan x \cot y) = 0$

On integrating, we obtain

$$\frac{3}{2}\tan^2 x-7\tan x+\frac{7}{2}\cot^2 y-5\cot y+4\tan x\cdot\cot y=c$$

42. (a) : Here $V_0 = 100$, $a = 20$, $b = 0$, and $e = f = 5$

$$\text{Hence } \frac{dQ}{dt}+\frac{1}{20}Q=0$$

The solution of this linear equation is $Q = ce^{-t/20}$ ...(i)

At $t = 0$, we are given that $Q = a = 20$

Substituting these values into equation (i) we find that $c = 20$, so that equation (i) can be rewritten as $Q = 20e^{-t/20}$. For $t = 20$, $Q = 20/e$

43. (b) : Here $a = 0$, $b = 1$, $e = 4$, $f = 2$, and $V_0 = 10$

The volume of brine in the tank at any time t is given as $V_0 + et - ft = 10 + 2t$

We require t when $10 + 2t = 50$, hence, $t = 20$ min.

44. (d) : For the equation $\frac{dQ}{dt}+\frac{2}{10+2t}Q=4$

This is a linear equation; its solution is

$$Q=\frac{40t+4t^2+c}{10+2t} \qquad \text{...(i)}$$

At $t = 0$, $Q = a = 0$. Substituting these values into equation (i), we find that $c = 0$. We require Q at the moment of overflow, *i.e.* $t = 20$.

$$\text{Thus, } Q=\frac{40(20)+4(20)^2}{10+2(20)}=48\,\text{lb}$$

45. (b) : $\because\ \frac{dT}{dt}=-k(T-290) \Rightarrow \frac{dT}{(T-290)}=-k\,dt$

On integrating, we get

$\Rightarrow$ $\ln(T - 290) = -kt + c$ ...(i)

Initially, $T = 370$ K and $t = 0$, then

$\ln(80) = c$ ...(ii)

From Eq. (i) and (ii), $\ln(T - 290) = -kt + \ln 80$

$$\ln\left(\frac{T-290}{80}\right)=-kt \Rightarrow \frac{T-290}{80}=e^{-kt}$$

or $T = 290 + 80e^{-kt}$

46. (d) : For t =10 min, $T = 330$

Then, $330 - 290 = 80e^{-10k}$

$$\Rightarrow\ \frac{1}{2}=e^{-10k} \Rightarrow e^{10k}=2 \Rightarrow 10k=\ln 2 \text{ or } k=\frac{\ln 2}{10}$$

47. (c) : At $T = 295$ K

$295 - 290 = 80e^{-kt} \Rightarrow 16 = e^{kt}$

$$\Rightarrow\ 4\ln 2=kt=\frac{\ln 2}{10}\times t\ \therefore\ t=40 \text{ min}$$

48. A – P, S; B – P, S; C – Q, R

49. A – Q ; B – R ; C – P

A. $y = e^{4x} + 2e^{-x}; y_1 = 4e^{4x} - 2e^{-x};$

$y_2 = 16e^{4x} + 2e^{-x}; y_3 = 64e^{4x} - 2e^{-x}$

Now, $y_3 - 13y_1 = (64e^{4x} - 2e^{-x}) - 13(4e^{4x} - 2e^{-x})$

$= 12e^{4x} + 24e^{-x} \Rightarrow y_3 - 13y = 12(e^{4x} + 2e^{-x}) = 12y$

$\therefore$ $K = 12$ and $K/3 = 4$

B. Since equation is of 2 degree, two lines are possible.

C. $y = u^m \Rightarrow \dfrac{dy}{dx} = m u^{m-1} \dfrac{du}{dx}$

Substituting the value of y and $\dfrac{dy}{dx}$

in $2x^4 y \dfrac{dy}{dx} + y^4 = 4x^6$ we have

$$2x^4 u^m m u^{m-1} \frac{du}{dx} + u^{4m} = 4x^6 \Rightarrow \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2m x^4 u^{2m-1}}$$

For homogeneous equation, $4m = 6 \Rightarrow m = \dfrac{3}{2}$

and $2m - 1 = 2 \Rightarrow m = \dfrac{3}{2}$

50. (3)

51. (2) : Put $x = \dfrac{1}{X}$ and $y = \dfrac{1}{Y}$

$dx = -\dfrac{1}{X^2} dX$ and $dy = -\dfrac{1}{Y^2} dY$

$$\Rightarrow \quad p = \frac{dy}{dx} = \frac{X^2}{Y^2}\frac{dY}{dX} = \frac{X^2}{Y^2} P \quad \left(\frac{dY}{dX} = P \text{ (say)}\right)$$

$\therefore$ The given equation becomes

$$\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X} \cdot \frac{X^2}{Y^2} P\right) = \frac{1}{X^4} \cdot \frac{X^4}{Y^4} P^2$$

$\Rightarrow$ $Y - XP = P^2$ or $Y = PX + P^2$

Which is the Clairaut's form

$\therefore$ The solution is $Y = cX + c^2$ or $\dfrac{1}{y} = \dfrac{c}{x} + c^2$

52. (3) : The given equation is $y = 2px + y^2p^3$ ...(i)

Solving for x, $x = \dfrac{y}{2p} - \dfrac{1}{2} y^2 p^2$

Differentiating w.r.t y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2} \cdot \frac{dp}{dy} - yp^2 - y^2 p \cdot \frac{dp}{dy}$$

or $2p = p - y\dfrac{dp}{dy} - 2yp^4 - 2y^2p^3 \dfrac{dp}{dy}$

or $p(1 + 2yp^3) + y\dfrac{dp}{dy}(1 + 2yp^3) = 0$

or $(1 + 2yp^3)\left(p + y\dfrac{dp}{dy}\right) = 0$

Neglecting the first factor which does not involve $\dfrac{dp}{dy}$, we have

$p + y\dfrac{dp}{dy} = 0 \quad \Rightarrow \dfrac{dp}{p} + \dfrac{dy}{y} = 0$

On integrating we get, $\log p + \log y = \log c$

or $\log py = \log c \Rightarrow py = c$ ...(ii)

Eliminating p between (i) and (ii)

$y = 2x \cdot \dfrac{c}{y} + y^2 \cdot \dfrac{c^3}{y^3}$ or $y = \dfrac{2cx}{y} + \dfrac{c^3}{y}$

or $y^2 = 2cx + c^3$ which is the required solution

53. (4) : Put $\sin y = t \Rightarrow \cos y \dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\therefore$ $\dfrac{dt}{dx} + \dfrac{4}{x} t = \dfrac{e^x}{x^3}$

$\text{I.F.} = e^{\int (4/x)dx} = e^{4\ln x} = x^4$

$\therefore$ The solution is $tx^4 = \int xe^x dx + \lambda$

$\Rightarrow$ $tx^4 = e^x(x - 1) + \lambda$ *i.e.*, $\sin y = e^x(x - 1)x^{-4}$

$\lambda = 0$ due to initial condition.

54. (3)

55. (1) : Equation of normal at the point $p(x, y)$ is

$$Y - y = -\frac{dx}{dy}(X - x)$$

Let, $m = \dfrac{dy}{dx} \Rightarrow X + mY - (x + my) = 0$ ...(i)

Distance of perpendicular from the origin to line (i)

is $\dfrac{|x + my|}{\sqrt{1 + m^2}} = |y| \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$

This is homogeneous equation. Let, $y = zx$

$\Rightarrow \dfrac{dy}{dx} = z + x\dfrac{dz}{dx} \Rightarrow \dfrac{2z}{1 + z^2} dz = -\dfrac{dx}{x}$

Integrating both sides, $\int \dfrac{2z}{1 + z^2} dz = -\int \dfrac{dx}{x}$

$\Rightarrow$ $\log(1 + z^2) = -\log x + c \Rightarrow (x^2 + y^2) = x \cdot e^c$

This curve passes through (1, 1)

$\Rightarrow$ $1 + 1 = 1 \cdot e^c \Rightarrow e^c = 2$

The required equation of the curve is

$\Rightarrow$ $x^2 + y^2 = 2x$

MPP-5 CLASS XI ANSWER KEY

1.	(a)	**2.**	(d)	**3.**	(b)	**4.**	(d)	**5.**	(a)
6.	(a)	**7.**	(a,d)	**8.**	(a,b,c,d)	**9.**	(a,b,c,d)		
10.	(b,d)	**11.**	(b,c)	**12.**	(a,b,c)	**13.**	(b)	**14.**	(b)
15.	(d)	**16.**	(b)	**17.**	(8)	**18.**	(3)	**19.**	(0)
20.	(1)								

CLASS XII **Series 7**

ACE YOUR WAY CBSE

Vector Algebra

HIGHLIGHTS

VECTOR

Quantities which have both magnitude and direction are called vector.

$A \longrightarrow B$

It is generally represented by $\overrightarrow{AB}$, where A is the initial point and B is the terminal point.
The arrow indicates the direction of the vector.

MAGNITUDE OF A VECTOR

The magnitude of a vector $\overrightarrow{AB}$ is the length of the line segment representing it. So, magnitude will be non-negative quantity. It is represented by $|\overrightarrow{AB}|$.

POSITION VECTOR

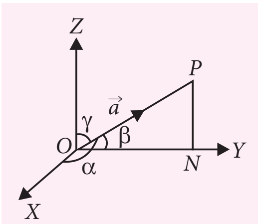
If P is a point in space having coordinates (x, y, z) w.r.t. the origin $O(0, 0, 0)$. Then the vector $\overrightarrow{OP}$ is called the position vector of the point P. The magnitude of $\overrightarrow{OP}$ is,

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

Previous Years Analysis						
	2016		2015		2014	
	Delhi	AI	Delhi	AI	Delhi	AI
VSA	2	2	2	2	2	2
SA	1	1	1	1	1	1
LA	-	-	-	-	-	-

DIRECTION COSINES

If a vector $\vec{a}$ makes angles α, β, γ with the positive direction of X, Y and Z axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines of vector $\vec{a}$ and are usually denoted by l, m, n.



Types of Vectors

Zero or Null Vector
A vector whose magnitude is 0 and direction is in determinate.
Representation : $\vec{0}$

Unit Vector
A vector whose magnitude is unity.
Representation : $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Coinitial Vectors
Vectors with same initial point.

Collinear or Parallel Vectors
Vectors parallel to the same line irrespective of their magnitudes and directions.

Equal Vectors
Vectors having equal magnitudes and same direction.

Negative of a Vector
Vectors having same magnitude but opposite in direction.

Unlike Vectors
Vectors having opposite direction.

Like Vectors
Vectors having same direction.

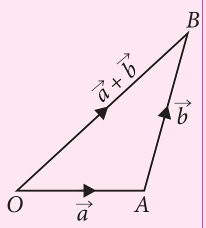
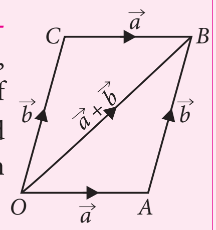
Coplanar Vectors
Vectors lie in the same plane or parallel to the same plane.

COMPONENTS OF A VECTOR

Let O be the origin and $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along X, Y and Z-axes respectively. Then $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Here x, y, z are scalar components and $x\hat{i}, y\hat{j}, z\hat{k}$ are vector components of $\overrightarrow{OP}$.

OPERATIONS ON VECTORS

Operations		Properties
Addition of two vectors	**Triangle Law** - In ΔOAB, let position vectors of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ be $\vec{a}$ and $\vec{b}$ respectively. Then $\overrightarrow{OB} = \vec{a} + \vec{b}$ 	**Commutative Law** - $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ **Associative Law** - $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$ **Existence of Additive Identity** - $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ *i.e.*, $\vec{0}$ is the additive identity **Existence of Additive Inverse** - $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$ *i.e.*, $(-\vec{a})$ is the additive inverse of $\vec{a}$.
	Parallelogram Law - In parallelogram $OABC$, let position vectors of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ be $\vec{a}$ and $\vec{b}$ respectively. Then $\overrightarrow{OB} = \vec{a} + \vec{b}$ 	
Multiplication of a vector by a scalar	Let $\vec{a}$ be any vector. Then for any scalar m, the product $m\vec{a}$ is defined as the vector whose magnitude is $\lvert m \rvert$ times to that of $\vec{a}$ and the direction is same as that of $\vec{a}$, if m is positive and opposite to that of $\vec{a}$ if m is negative.	**Commutative Law** - $m(\vec{a})=(\vec{a})m=m\vec{a}$ **Associative Law** - $m(n\vec{a})=n(m\vec{a})=mn(\vec{a})$ **Distributive Law** - $(m+n)\vec{a}=m\vec{a}+n\vec{a}$ and $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$
Multiplication of two vectors	**Scalar Product or Dot Product** - Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors, then scalar or dot product of $\vec{a}$ and $\vec{b}$ is denoted and defined by, $\vec{a}\cdot\vec{b}=\lvert\vec{a}\rvert\,\lvert\vec{b}\rvert\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.	**Commutative Law** - $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ **Distributive Law** - $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
	Vector Product or Cross Product - Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors, then vector or cross product of $\vec{a}$ and $\vec{b}$ is denoted and defined by, $\vec{a}\times\vec{b}=\lvert\vec{a}\rvert\,\lvert\vec{b}\rvert\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector along $\vec{a}\times\vec{b}$.	**Distributive Law** - $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ • Vector product is not commutative. • Vector product is not associative.

SOME IMPORTANT RESULTS

Scalar Product	Vector Product
• $\vec{a}\cdot\vec{b}$ is a scalar quantity. • $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$ or $\vec{a}=0$ or $\vec{b}=0$ • If $\theta=0$, then $\vec{a}\cdot\vec{b}=\lvert\vec{a}\rvert\lvert\vec{b}\rvert$ • If $\theta=\pi$, then $\vec{a}\cdot\vec{b}=-\lvert\vec{a}\rvert\lvert\vec{b}\rvert$ • $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$ $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=\vec{0}$ • Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}$ or $\theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}\right)$	• $\vec{a}\times\vec{b}$ is a vector quantity. • $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \vec{a}\,\vert\vert\,\vec{b}$ • If $\theta=\frac{\pi}{2}$, then $\vec{a}\times\vec{b}=\lvert\vec{a}\rvert\lvert\vec{b}\rvert$ • $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$ $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$ • $\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j}$ • Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is $\sin\theta=\frac{\lvert\vec{a}\times\vec{b}\rvert}{\lvert\vec{a}\rvert\lvert\vec{b}\rvert}$ • Area of $\Delta ABC=\frac{1}{2}\lvert\vec{a}\times\vec{b}\rvert$, where $\vec{a},\vec{b}$ are adjacent sides. • Area of parallelogram $ABCD=\lvert\vec{a}\times\vec{b}\rvert$, where $\vec{a},\vec{b}$ are adjacent sides.

VECTOR JOINING TWO POINTS

Let $A\equiv(x_1, y_1, z_1)$ and $B\equiv(x_2, y_2, z_2)$. Then

$$\overrightarrow{AB}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

Also, $\lvert\overrightarrow{AB}\rvert=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

SECTION FORMULA

Let A and B be any two points with position vectors $\vec{a}$ and $\vec{b}$ respectively. Let $C(\vec{c})$ divides AB in the ratio $m:n$, then

- **Internal division :** $\vec{c}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$
- **External division :** $\vec{c}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$

MID POINT FORMULA

Let C is the mid point of AB then, $\vec{c}=\dfrac{\vec{a}+\vec{b}}{2}$

PROJECTION OF A VECTOR ON A LINE

- Projection of a given vector $\vec{a}$ along a directed line l is given by $\dfrac{\vec{a}\cdot\vec{b}}{\lvert\vec{p}\rvert}$, where $\vec{p}$ is a vector along the direction of the line.
- Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is given by $\vec{a}\cdot\hat{b}$ or $\vec{a}\cdot\dfrac{(\vec{b})}{\lvert\vec{b}\rvert}$ or $\dfrac{(\vec{a}\cdot\vec{b})}{\lvert\vec{b}\rvert}$

SCALAR TRIPLE PRODUCT

Let $\vec{a},\vec{b},\vec{c}$ be any three vectors. Then the scalar product of $\vec{a}\times\vec{b}$ and $\vec{c}$ is called scalar triple product of $\vec{a},\vec{b}$ and $\vec{c}$. It is denoted by $(\vec{a}\times\vec{b})\cdot\vec{c}$ or $[\vec{a}\ \vec{b}\ \vec{c}]$ or $[\vec{a},\vec{b},\vec{c}]$.

Remarks -

- $\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$,

  where $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$
- $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$
- $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]$
- $[\vec{a}\ \vec{a}\ \vec{b}]=0$
- Volume of parallelopied whose coterminous edges are $\vec{a},\vec{b},\vec{c}$ is $[\vec{a}\cdot(\vec{b}\times\vec{c})]$

COPLANARITY OF THREE VECTORS

Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ are said to be coplanar iff $\vec{a}\cdot(\vec{b}\times\vec{c})=0$ *i.e.*, $[\vec{a}\ \vec{b}\ \vec{c}]=0$

PROBLEMS

Very Short Answer Type

1. If $\vec{r}_1=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{r}_2=2\hat{i}-4\hat{j}-3\hat{k}$ and $\vec{r}_3=-\hat{i}+2\hat{j}+2\hat{k}$, find the modulus of $2\vec{r}_1-3\vec{r}_2-5\vec{r}_3$.

2. Let $\vec{a} = 2\hat{i} - 3\hat{j}$ and $\vec{b} = 3\hat{i} + 2\hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal ?

3. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 10$, $|\vec{b}| = 15$ and $\vec{a}\cdot\vec{b} = 75\sqrt{2}$, find the angle between $\vec{a}$ and $\vec{b}$.

4. Find a unit vector in the direction of the resultant of vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, $-\hat{i} + 2\hat{j} + \hat{k}$ and $3\hat{i} + \hat{j}$.

5. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $(\vec{a}\times\vec{b}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

Short Answer Type

6. Vectors drawn from the origin O to the points A, B and C are respectively $\vec{a}$, $\vec{b}$ and $4\vec{a} - 3\vec{b}$. Find $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

7. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other.

8. Find the projection of the vector $2\hat{i} - 3\hat{j} - 6\hat{k}$ on the line joining the points (5, 6, – 3) and (3, 4, – 2).

9. If $\vec{a} = 4\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{k}$, find $|\vec{b}\times 2\vec{a}|$.

10. Find the area of the parallelogram whose adjacent sides are $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Long Answer Type-I

11. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

12. If $\vec{a} = \hat{i} - \hat{j} - 3\hat{k}$, $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$, verify that $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$.

13. Prove that $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

14. **(i)** Find the volume of the parallelopiped whose coterminous edges are represented by the vectors $\vec{v}_1 = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{v}_2 = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v}_3 = 3\hat{i} - \hat{j} + 2\hat{k}$.

(ii) Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

15. Prove that $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = \vec{0}$.

Long Answer Type-II

16. Prove that four points $4\hat{i} + 5\hat{j} + \hat{k}$, $-(\hat{j} + \hat{k})$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ are coplanar.

17. Using vector method, prove that in a ΔABC,
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
where a, b, c are the lengths of the sides opposite to the angles A, B and C respectively of ΔABC.

18. **(i)** If $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = \vec{0}$, prove that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

(ii) Prove that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$.

19. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ show that the vectors $(\vec{a}-\vec{b})$, $(\vec{b}-\vec{c})$, $(\vec{c}-\vec{a})$ are coplanar.

20. Using vectors find the area of ΔABC whose vertices are $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

SOLUTIONS

1. $2\vec{r}_1 - 3\vec{r}_2 - 5\vec{r}_3$
$$= 2(3\hat{i} - 2\hat{j} + \hat{k}) - 3(2\hat{i} - 4\hat{j} - 3\hat{k}) - 5(-\hat{i} + 2\hat{j} + 2\hat{k})$$
$$= 5\hat{i} - 2\hat{j} + \hat{k}$$
$$\therefore\ |2\vec{r}_1 - 3\vec{r}_2 - 5\vec{r}_3| = \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{30}.$$

2. $|\vec{a}| = |2\hat{i} - 3\hat{j}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

and $|\vec{b}| = |3\hat{i} + 2\hat{j}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ $\therefore$ $|\vec{a}| = |\vec{b}|$

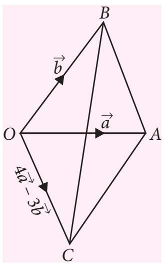
Since the corresponding components of the given vectors $\vec{a}$ and $\vec{b}$ are not equal therefore, $\vec{a} \neq \vec{b}$.

3. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, then $0 \le \theta \le \pi$ and $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$$\Rightarrow\ 75\sqrt{2} = 10\times 15\cos\theta \Rightarrow \cos\theta = \frac{75\sqrt{2}}{150}$$
$$\Rightarrow\ \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°.$$

4. Let $\vec{a}$ be the resultant of given vectors. Then,
$$\vec{a} = (\hat{i} + 2\hat{j} + 3\hat{k}) - \hat{i} + 2\hat{j} + \hat{k} + (3\hat{i} + \hat{j}) = 3\hat{i} + 5\hat{j} + 4\hat{k}$$
$$\therefore\ |\vec{a}| = \sqrt{3^2 + 5^2 + 4^2} = 5\sqrt{2}$$
Now unit vector along $\vec{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{3}{5\sqrt{2}}\hat{i} + \dfrac{5}{5\sqrt{2}}\hat{j} + \dfrac{4}{5\sqrt{2}}\hat{k}$

5. Given, $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a}\times\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$
$$\therefore\ |\vec{a}\times\vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$$
Let θ be the angle between $\vec{a}$ and $\vec{b}$
$$\text{Now, } \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{7}{2\times 7} = \frac{1}{2}\ \therefore\ \theta = \frac{\pi}{6}$$

6. In ΔOAC, we have $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$
$$\Rightarrow\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= 4\vec{a} - 3\vec{b} - \vec{a} = 3(\vec{a} - \vec{b})$$
In ΔOBC, we have
$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$$
$$\Rightarrow\ \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= 4\vec{a} - 3\vec{b} - \vec{b} = 4(\vec{a} - \vec{b}).$$



7. We have, $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$,

$\therefore\quad \vec{a}+\vec{b}=4\hat{i}+\hat{j}-\hat{k}$ and $\vec{a}-\vec{b}=-2\hat{i}+3\hat{j}-5\hat{k}$

$\therefore\quad (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=4(-2)+(1\times3)+(-1)(-5)=0.$

Thus the dot product of two non-zero vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is zero.

Therefore, the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular to each other.

8. Let $\vec{a}=2\hat{i}-3\hat{j}-6\hat{k}$, $A\equiv(5, 6, -3)$, $B\equiv(3, 4, -2)$

$\therefore\quad \overrightarrow{AB}=(3-5)\hat{i}+(4-6)\hat{j}+(-2+3)\hat{k}$

$=-2\hat{i}-2\hat{j}+\hat{k}$

Now the projection of $\vec{a}$ on $\overrightarrow{AB}$

$$=\frac{\vec{a}\cdot\overrightarrow{AB}}{|\overrightarrow{AB}|}=\frac{-4+6-6}{\sqrt{4+4+1}}=-\frac{4}{3}$$

9. $2\vec{a}=8\hat{i}+6\hat{j}+4\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{k}$

$\therefore\quad \vec{b}=3\hat{i}+2\hat{k}$ and $2\vec{a}=8\hat{i}+6\hat{j}+4\hat{k}$

Now, $\vec{b}\times2\vec{a}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&0&2\\8&6&4\end{vmatrix}=-12\hat{i}+4\hat{j}+18\hat{k}$

$\therefore\quad |\vec{b}\times2\vec{a}|=\sqrt{(-12)^2+4^2+(18)^2}=22$

10. Area of the parallelogram $=|\vec{a}\times\vec{b}|$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&3\\3&-2&1\end{vmatrix}=8\hat{i}+8\hat{j}-8\hat{k}$

$\therefore\quad |\vec{a}\times\vec{b}|=\sqrt{8^2+8^2+(-8)^2}=8\sqrt{3}$ sq. units.

11. Let $\vec{a}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{b}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

Then, $\vec{a}+\vec{b}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

Unit vector along the sum of these vectors is

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}$$

Given, $(\hat{i}+\hat{j}+\hat{k})\cdot\dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=1$

$\Rightarrow\quad 1\times(2+\lambda)+1\times6+1\times(-2)=\sqrt{(2+\lambda)^2+40}$

$\Rightarrow\quad \lambda+6=\sqrt{(2+\lambda)^2+40}\Rightarrow\lambda=1.$

12. Given, $\vec{a}=\hat{i}-\hat{j}-3\hat{k}, \vec{b}=4\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+2\hat{k}$

Now, $\vec{b}+\vec{c}=(4\hat{i}-3\hat{j}+\hat{k})+(2\hat{i}-\hat{j}+2\hat{k})$

$=6\hat{i}-4\hat{j}+3\hat{k}$

$\therefore\quad \vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&-3\\6&-4&3\end{vmatrix}=-15\hat{i}-21\hat{j}+2\hat{k}$ ...(i)

Also, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&-3\\4&-3&1\end{vmatrix}=-10\hat{i}-13\hat{j}+\hat{k}$

and $\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&-3\\2&-1&2\end{vmatrix}=-5\hat{i}-8\hat{j}+\hat{k}$

$\therefore\quad \vec{a}\times\vec{b}+\vec{a}\times\vec{c}=(-10\hat{i}-13\hat{j}+\hat{k})+(-5\hat{i}-8\hat{j}+\hat{k})$

$=-15\hat{i}-21\hat{j}+2\hat{k}$ ...(ii)

From (i) and (ii), we have

$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

13. L.H.S. $=[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]$

$=(\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$=(\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$

$=(\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]\quad[\because\ \vec{c}\times\vec{c}=\vec{0}]$

$=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})$
$\quad+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})$

$=[\vec{a}\;\vec{b}\;\vec{c}]+[\vec{a}\;\vec{b}\;\vec{a}]+[\vec{a}\;\vec{c}\;\vec{a}]+[\vec{b}\;\vec{b}\;\vec{c}]$
$\quad+[\vec{b}\;\vec{b}\;\vec{a}]+[\vec{b}\;\vec{c}\;\vec{a}]$

$=[\vec{a}\;\vec{b}\;\vec{c}]+[\vec{b}\;\vec{c}\;\vec{a}]=2[\vec{a}\;\vec{b}\;\vec{c}]=$ R.H.S

14. **(i)** $[\vec{v}_1\;\vec{v}_2\;\vec{v}_3]=\begin{vmatrix}2&1&-1\\1&2&3\\3&-1&2\end{vmatrix}=14+7+7=28.$

Hence, the required volume = 28 cubic units.

(ii) Given vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if $[\vec{a}\;\vec{b}\;\vec{c}]=0$

Now, $[\vec{a}\;\vec{b}\;\vec{c}]=\begin{vmatrix}1&-2&3\\-2&3&-4\\1&-3&5\end{vmatrix}=3-12+9=0$

Hence, the given vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

15. We have,

$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})+(\vec{c}\times\vec{b})$

[by distributive law]

$=(\vec{a}\times\vec{b})-(\vec{c}\times\vec{a})+(\vec{b}\times\vec{c})-(\vec{a}\times\vec{b})+(\vec{c}\times\vec{a})-(\vec{b}\times\vec{c})$

$=\vec{0}$

Hence, $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}.$

16. Let the given points be A, B, C, D respectively.

Let O be the origin, then

$$\overrightarrow{OA} = 4\hat{i} + 5\hat{j} + \hat{k},\ \overrightarrow{OB} = -(\hat{j} + \hat{k}),$$

$$\overrightarrow{OC} = 3\hat{i} + 9\hat{j} + 4\hat{k},\ \overrightarrow{OD} = 4(-\hat{i} + \hat{j} + \hat{k})$$

$$\therefore\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -(\hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$$

$$= -4\hat{i} - 6\hat{j} - 2\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$$

$$= -\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 4(-\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$$

$$= -8\hat{i} - \hat{j} + 3\hat{k}$$

Now, $[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$

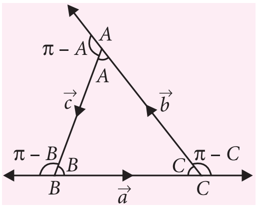
$= -60 + 126 - 66 = 0$

Hence the points A, B, C, D are coplanar.

17. In triangle ABC

Let $\overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}, \overrightarrow{AB} = \vec{c}$

then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ ...(i)



Taking vector product with $\vec{a}$, on both sides of (i) we have $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

$$\Rightarrow\quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow\quad \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0} \Rightarrow \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$$

$$\Rightarrow\quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \qquad \text{...(ii)}$$

Similarly, taking vector product with $\vec{b}$ on both sides of (i) we have

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \qquad \text{...(iii)}$$

From (ii) and (iii), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$$\Rightarrow\quad |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

or $ab \sin (\pi - C) = bc \sin (\pi - A) = ca \sin (\pi - B)$

or $ab \sin C = bc \sin A = ca \sin B$

Dividing throughout by abc, we get

$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b} \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

18. **(i)** Let $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$.

$\Rightarrow\quad \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$

and $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

[$\because \vec{a} \perp \vec{b}$ and $\vec{a} \parallel \vec{b}$ can never hold simultaneously].

Hence, $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$

$\Rightarrow\quad \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

(ii) We have $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$ [by distributive law]

$= \vec{a} \times \vec{b} - \vec{b} \times \vec{a}$ [$\because \vec{a} \times \vec{a} = \vec{0}$ and $\vec{b} \times \vec{b} = \vec{0}$]

$= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) = 2(\vec{a} \times \vec{b})$ [$\because -(\vec{b} \times \vec{a}) = (\vec{a} \times \vec{b})$]

Hence, $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

19. Consider, $[\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}]$

$= (\vec{a} - \vec{b}) \cdot [(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})]$

$= (\vec{a} - \vec{b}) \cdot [\vec{b} \times \vec{c} - (\vec{b} \times \vec{a}) - (\vec{c} \times \vec{c}) + \vec{c} \times \vec{a}]$

[by distributive law]

$= (\vec{a} - \vec{b}) \cdot [\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}]$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot (\vec{c} \times \vec{a})$

$\quad - \vec{b} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{a} \times \vec{b}) - \vec{b} \cdot (\vec{c} \times \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{a}\ \vec{b}] + [\vec{a}\ \vec{c}\ \vec{a}] - [\vec{b}\ \vec{b}\ \vec{c}]$

$\quad - [\vec{b}\ \vec{a}\ \vec{b}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [a\ \vec{b}\ \vec{c}] = 0$ ($\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$)

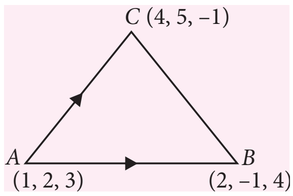
Hence, the vectors $(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ are coplanar.

20. We have,

Position vector of $A = (\hat{i} + 2\hat{j} + 3\hat{k})$,

Position vector of $B = (2\hat{i} - \hat{j} + 4\hat{k})$ and

Position vector of $C = (4\hat{i} + 5\hat{j} - \hat{k})$



$$\therefore\quad \overrightarrow{AB} = (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} - 3\hat{j} + \hat{k}$$

$$\overrightarrow{AC} = (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} - 4\hat{k}$$

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$

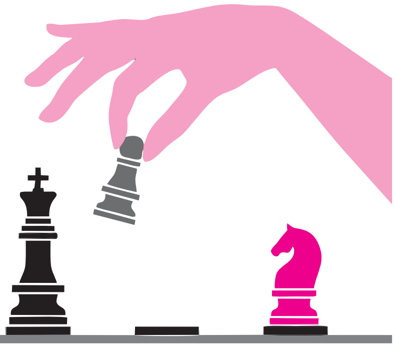
$= (12 - 3)\hat{i} - (-4 - 3)\hat{j} + (3 + 9)\hat{k} = 9\hat{i} + 7\hat{j} + 12\hat{k}$

$$\therefore\quad \text{Area of } \Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(9\hat{i} + 7\hat{j} + 12\hat{k})| = \frac{1}{2} \cdot \sqrt{(9)^2 + (7)^2 + (12)^2}$$

$$= \frac{1}{2} \cdot \sqrt{(81 + 49 + 144)} = \frac{1}{2}\sqrt{274} \text{ sq. units}$$

Challenging PROBLEMS ON Vectors and 3D Geometry



1. Vertex A of the acute triangle ABC is equidistant from the circumcentre O and the orthocentre H. The possible value of measure of $\angle A$ is
(a) 30° (b) 60° (c) 75° (d) 90°

2. If A, B, C, D are four points in space such that $\angle ABC = \angle BCD = \angle CDA = \angle DAB = \pi/2$, then
(a) A, B, C, D are coplanar points
(b) A, B, C, D are non-coplanar points
(c) line AB is skew to line CD
(d) nothing definite can be said about their relative positions

3. Given P, Q, R, S are points on the sides of the quadrilateral $ABCD$ such that $\frac{AP}{PB}=\frac{BQ}{QC}=\frac{CR}{RD}=\frac{DS}{SA}=k$ and the area of the quadrilateral $PQRS$ is exactly 52% of the area of quadrilateral $ABCD$ then $k =$
(a) 1/3 (b) 2/3 (c) 2/1 (d) 1/2

4. The volume of the region in space defined by $|x + y + z| + |x + y - z| \le 8$, $(x, y, z) \ge 0$ is
(a) 16 (b) 24 (c) 32 (d) 48

5. Let P be a point in the interior of a tetrahedron $ABCD$ such that its projections A_1, B_1, C_1, D_1 onto the planes (BCD), (CDA), (DAB), (ABC) respectively are all situated in the interior of the faces. If Δ is the total area and r, the inradius of the tetrahedron, such that $\frac{\Delta_{BCD}}{PA_1}+\frac{\Delta_{CDA}}{PB_1}+\frac{\Delta_{DAB}}{PC_1}+\frac{\Delta_{ABC}}{PD_1}=\frac{\Delta}{r}$ then point P is
(a) centroid of tetrahedron $ABCD$
(b) circumcentre of tetrahedron $ABCD$
(c) incentre of tetrahedron $ABCD$
(d) orthocentre of tetrahedron $ABCD$

6. If 'a' is a real constant and α, β, γ are variable angles and $(\sqrt{a^2-4})\tan\alpha + a\tan\beta + (\sqrt{a^2+4})\tan\gamma = 6a$ then the least value of $\tan^2\alpha + \tan^2\beta + \tan^2\gamma$ is
(a) 3 (b) 4 (c) 7 (d) 12

7. If vectors $\vec{a} = (c\log_2 x)\hat{i} - 6\hat{j} + 3\hat{k}$ and $\vec{b} = (\log_2 x)\hat{i} + 2\hat{j} + (2c\log_2 x)\hat{k}$ make an obtuse angle for any $x \in (0, \infty)$ then $c \in$
(a) $\left(-\frac{4}{3}, 0\right)$ (b) $(-2, 0)$
(c) $\left(-\frac{4}{3}, 1\right)$ (d) $\left(-2, -\frac{4}{3}\right)$

8. The maximum value of $|\vec{a}-\vec{b}|^2 + |\vec{b}-\vec{c}|^2 + |\vec{c}-\vec{a}|^2$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors is
(a) 6 (b) 9 (c) 12 (d) 24

9. If $\hat{v}$ is the unit vector along the incident ray, $\hat{w}$ is the unit vector along the reflected ray and $\hat{a}$ is the unit vector along the outward normal to the plane mirror at the point of incidence then $\hat{w} =$
(a) $\hat{v} - 2\hat{a}(\hat{a}\cdot\hat{v})$ (b) $\hat{v} - \hat{a}(\hat{a}\cdot\hat{v})$
(c) $\hat{a} - 2\hat{v}(\hat{a}\cdot\hat{v})$ (d) $\hat{a} + 2\hat{v}(\hat{a}\cdot\hat{v})$

10. Arc AC of a circle subtends a right angle at the centre O. Point B divides the arc in the ratio 1 : 2. If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ then $\overrightarrow{OC} =$
(a) $\sqrt{3}\vec{a} + 2\vec{b}$ (b) $-\sqrt{3}\vec{a} + 2\vec{b}$
(c) $\sqrt{3}\vec{a} - 2\vec{b}$ (d) $2\vec{a} - 3\vec{b}$

11. If (l, m, n) are the direction cosines of a line, then maximum value of product (lmn) is
(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $3\sqrt{3}$ (d) $\frac{1}{3\sqrt{3}}$

12. The four planes $lx + my = 0$, $nz + lx = 0$, $my + nz = 0$ and $lx + my + nz = p$ form a tetrahedron whose volume is
(a) $\frac{2p^3}{3lmn}$ (b) $\frac{3p^3}{2lmn}$ (c) $\frac{3lmn}{2p^3}$ (d) $\frac{2lmn}{3p^3}$

13. If the lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular then
(a) $aa' + cc' + 1 = 0$ (b) $aa' - bb' = 0$
(c) $aa' = dd'$ (d) $bc' + b'c + 1 = 0$

By : Tapas Kr. Yogi, Mob : 9533632105.

14. A uni-modular tangent vector on the curve $x = t^2 + 2, y = 4t - 5, z = 2t^2 - 6t$ at $t = 2$ is $\lambda(2\hat{i} + 2\hat{j} + \hat{k})$ where $\lambda =$

(a) 1 (b) 1/2 (c) 1/3 (d) 1/4

15. A rigid body rotates about an axis through the origin with an angular velocity of $10\sqrt{3}$ radians/sec. If $\vec{\omega}$ point in the direction of $\hat{i} + \hat{j} + \hat{k}$ then the equation to the locus of the points having tangential speed of 20 m/sec is $x^2 + y^2 + z^2 - xy - yz - zx + \lambda = 0$ where $\lambda =$

(a) 1 (b) −1 (c) 2 (d) −2

16. A rod of length 2 units whose one end is (1, 0, −1) and the other end moves on the plane $x - 2y + 2z + 4 = 0$ encloses a volume (in cubic units) of

(a) π (b) 2π (c) 3π (d) 4π

17. Let T be the largest of the areas of the four faces of a tetrahedron and Y be the total surface area of the tetrahedron. Which of the following statements is true?

(a) $2 < \frac{Y}{T} \le 4$ (b) $3 < \frac{Y}{T} \le 4$

(c) $\frac{5}{2} < \frac{Y}{T} \le \frac{9}{2}$ (d) $\frac{7}{2} < \frac{Y}{T} < \frac{11}{2}$

18. Given ΔABC and ΔAEF such that B is the midpoint of EF. Also $AB = EF = 1$, $BC = 6$, $CA = \sqrt{33}$ and $\overrightarrow{AB}\cdot\overrightarrow{AE} + \overrightarrow{AC}\cdot\overrightarrow{AF} = 2$. The cosine of the angle between $\overrightarrow{EF}$ and $\overrightarrow{BC}$ is

(a) 1/3 (b) 2/3 (c) 3/4 (d) 3/5

SOLUTIONS

1. (b) : Let circumcentre be origin in our vector system then $|\vec{A} - \vec{0}| = |\vec{A}| = |\vec{A} - \vec{H}|$ and circumradius $R = |\vec{A}| = |\vec{B}| = |\vec{C}|$ and $\vec{H} = \vec{A} + \vec{B} + \vec{C}$ and let $a = |\vec{B} - \vec{C}|$

$\Rightarrow R^2 = |\vec{A}|^2 = |\vec{A} - \vec{H}|^2 = |\vec{B} + \vec{C}|^2$

So, $R^2 = 4R^2 - a^2$

i.e., $\frac{a}{R} = \sqrt{3} \Rightarrow \sin A = \frac{\sqrt{3}}{2}$ (by sine rule).

So, $A = 60°$

2. (a) : Let $B = (0, 0, 0)$, $A = (v, 0, 0)$, $C = (0, w, 0)$, $D = (a, b, c)$.

Since, $\angle BAD = 90°$. So $a = v$ and $\angle BCD = \pi/2$ gives $b = w$.

Now, $v^2 + w^2 = CA^2 = CD^2 + DA^2$
$= v^2 + c^2 + w^2 + c^2$

$\Rightarrow c = 0$

i.e. Point D is also in the XY plane along with the other points; *i.e.* coplanar.

3. (b) : Using section formula,

$$\overrightarrow{PR} = \frac{k\overrightarrow{BD} + \overrightarrow{AC}}{k+1} \text{ and } \overrightarrow{QS} = \frac{-k\overrightarrow{AC} + \overrightarrow{BD}}{k+1}$$

So, area of $PQRS = [PQRS] = (\overrightarrow{PR} \times \overrightarrow{QS})\cdot\frac{1}{2}$

$$= \frac{1}{2}\left(\left(\frac{k\overrightarrow{BD} + \overrightarrow{AC}}{k+1}\right)\times\left(\frac{-k\overrightarrow{AC} + \overrightarrow{BD}}{k+1}\right)\right)$$

$$= \frac{(k^2+1)}{2(k+1)^2}\cdot(\overrightarrow{AC}\times\overrightarrow{BD})$$

$$[P\,Q\,R\,S] = \frac{(k^2+1)}{(k+1)^2}[ABCD]$$

$$\Rightarrow \frac{52}{100} = \frac{k^2+1}{(k+1)^2}$$

Solving, $k = \frac{2}{3}$ or $\frac{3}{2}$

4. (c): Let $a = x + y$, so, $|a + z| + |a - z| \le 8$

$\Rightarrow 8 \ge |a + z| + |a - z| \ge |a + z + a - z| = 2a$

i.e. $a = x + y \le 4$

and $8 \ge |a + z| + |a - z| \ge (a + z) - (a - z) = 2z$

i.e. $z \le 4$

So, $0 \le x + y \le 4$, $0 \le z \le 4$.

So, a right prism formed with vertices (0, 0), (4, 0), (0, 4) and height = 4.

So, volume = (base area) × height $= \frac{4\times4}{2}\times4 = 32$

5. (c): Using Cauchy-Schwarz inequality,

$$\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4}\right)(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)$$
$$\ge (a_1 + a_2 + a_3 + a_4)^2$$

With $a_1 = \Delta_{BCD}$, $a_2 = \Delta_{CDA}$, $a_3 = \Delta_{DAB}$, $a_4 = \Delta_{ABC}$, $b_1 = PA_1$, $b_2 = PB_1$, $b_3 = PC_1$, $b_4 = PD_1$ gives,

$3V\sum\frac{\Delta_{BCD}}{PA_1} \ge \Delta^2$, where V is volume.

So, $\sum\frac{\Delta_{BCD}}{PA_1} \ge \frac{\Delta^2}{3V} = \frac{\Delta^2}{r\Delta} = \frac{\Delta}{r}$

and equality occurs if $\sqrt{\frac{a_i}{b_i}} = \lambda\cdot\sqrt{a_ib_i}$ $(i = 1, 2, 3, 4)$

i.e. $b_i = \frac{1}{\lambda}$. So, $PA_1 = PB_1 = PC_1 = PD_1$

i.e. P is the incentre of tetrahedron $ABCD$.

6. (d) : Consider the two vectors

$\vec{P} = (\sqrt{a^2-4})\,\hat{i} + a\,\hat{j} + (\sqrt{a^2+4})\,\hat{k}$ and

$\vec{Q} = (\tan\alpha)\,\hat{i} + (\tan\beta)\,\hat{j} + (\tan\gamma)\,\hat{k}$ then

$(\vec{P}\cdot\vec{Q})^2 \le |\vec{P}|^2|\vec{Q}|^2$ gives the required least value = 12

7. (a) : $\vec{a}\cdot\vec{b}<0$ (obtuse angle)

$\Rightarrow$ $c\cdot(\log_2 x)^2+6c(\log_2 x)-12<0$

So, $c<0$ and $36c^2+48c<0$ *i.e.* $c\in\left(-\frac{4}{3},\ 0\right)$

8. (b) : Using, $|\vec{a}+\vec{b}+\vec{c}|^2\geq 0\ \Rightarrow\ \sum(\vec{a}\cdot\vec{b})\geq -3/2$

So, $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=2(3)-2\sum(\vec{a}\cdot\vec{b})=9$

Equality occurs when each of the vectors are inclined at 120° to each other.

9. (a) : Let θ be the incident angle to normal to the mirror then $|\hat{w}-\hat{v}|^2=|\hat{w}|^2+|\hat{v}|^2-2\hat{w}\cdot\hat{v}$

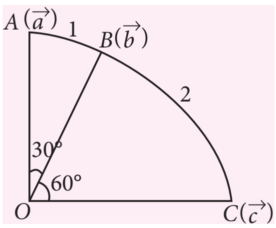
$$=1+1-2\cdot 1\cdot 1\cos(\pi-2\theta)$$

$$\Rightarrow\ |\hat{w}-\hat{v}|=2\cos\theta\ \text{ and }\ \hat{a}=\frac{\hat{w}-\hat{v}}{|\hat{w}-\hat{v}|}$$

So, $\hat{w}=\hat{v}-2\hat{a}(\hat{a}\cdot\hat{v})$

10. (b) : Let $\vec{c}=x\vec{a}+y\vec{b}$

$|\vec{a}|=|\vec{b}|=|\vec{c}|=r$



$$\Rightarrow\ \vec{a}\cdot\vec{b}=\frac{\sqrt{3}r^2}{2},\ \vec{a}\cdot\vec{c}=0$$

and $\vec{b}\cdot\vec{c}=\dfrac{r^2}{2}$

Considering $\vec{c}\cdot\vec{c}$ and simplifying we have

$$x=-\sqrt{3},\ y=2\ \ i.e.\ \ \vec{c}=-\sqrt{3}\vec{a}+2\vec{b}$$

11. (d) : Use A.M. ≥ G.M. on l^2, m^2, n^2 and $l^2+m^2+n^2=1$

12. (a) : On solving the four plane equations three at a time, we have the vertices of the tetrahedron as (0, 0, 0), $\left(\frac{p}{l},\frac{p}{m},\frac{-p}{n}\right),\left(\frac{-p}{l},\frac{p}{m},\frac{p}{n}\right),\left(\frac{p}{l},\frac{-p}{m},\frac{p}{n}\right)$

$$\text{So, required volume}=\frac{1}{6}\begin{vmatrix}0 & 0 & 0 & 1\\ p/l & p/m & -p/n & 1\\ -p/l & p/m & p/n & 1\\ p/l & -p/m & p/n & 1\end{vmatrix}$$

$$=\frac{2p^3}{3lmn}$$

13. (a) : Rewrite the given line equation as

$$\frac{x-b'}{a'}=\frac{y-0}{1}=\frac{z-d'}{c'}\ \text{ and }$$

$$\frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{c}$$

So, perpendicular condition gives

$$aa'+1\cdot 1+cc'=0.$$

14. (c): The position vector at any point t is

$$\vec{r}=(2+t^2)\hat{i}+(4t-5)\hat{j}+(2t^2-6t)\hat{k}$$

So, $\left.\dfrac{d\vec{r}}{dt}\right|_{t=2}=4\hat{i}+4\hat{j}+2\hat{k}$ and

$$\left|\left[\frac{d\vec{r}}{dt}\right]_{t=2}\right|=\sqrt{16+16+4}=6$$

So, unit tangent vector $=\dfrac{4\hat{i}+4\hat{j}+2\hat{k}}{6}$

15. (d) : $\hat{n}=\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$, $\vec{\omega}=|\vec{\omega}|\hat{n}=10(\hat{i}+\hat{j}+\hat{k})$

$$\vec{v}=\vec{\omega}\times\vec{r}=10(\hat{i}+\hat{j}+\hat{k})\times(x\hat{i}+y\hat{j}+z\hat{k})$$

and $|\vec{v}|=20$ gives locus as

$$x^2+y^2+z^2-xy-zx-yz-2=0.$$

16. (c): The rod sweeps out a cone

The distance of (1, 0, –1) from the plane is $\dfrac{|1-2+4|}{\sqrt{9}}=1$

The slant height of the cone = 2

So, radius of base $=\sqrt{4-1}=\sqrt{3}$

So, volume $=\pi(\sqrt{3})^2\cdot 1=3\pi$

17. (a) : Clearly $\dfrac{Y}{T}\leq 4$, with equality holding for regular tetrahedron. On the other hand since the sum of the areas of any three faces of a tetrahedron is greater than that of the fourth so, $\dfrac{Y}{T}>2$.

18. (b) : From given data,

$$2=\overrightarrow{AB}\cdot\overrightarrow{AE}+\overrightarrow{AC}\cdot\overrightarrow{AF}=\overrightarrow{AB}\cdot(\overrightarrow{AB}+\overrightarrow{BE})+\overrightarrow{AC}\cdot(\overrightarrow{AB}+\overrightarrow{BF})$$

i.e., $\overrightarrow{AB}^2+\overrightarrow{AB}\cdot\overrightarrow{BE}+\overrightarrow{AC}\cdot\overrightarrow{AB}+\overrightarrow{AC}\cdot\overrightarrow{BF}=2$ ...(1)

and $\overrightarrow{AB}^2=1,\ \overrightarrow{AC}\cdot\overrightarrow{AB}=\sqrt{33}\cdot 1\cdot\dfrac{(33+1-36)}{2\cdot\sqrt{33}\cdot 1}=-1$

and $\overrightarrow{BE}=-\overrightarrow{BF}$ so, eqn.(1) becomes

$$1+\overrightarrow{BF}\cdot(\overrightarrow{AC}-\overrightarrow{AB})-1=2\ \Rightarrow\ \overrightarrow{BF}\cdot\overrightarrow{BC}=2$$

i.e., $|\overrightarrow{BF}||\overrightarrow{BC}|\cos\theta=2$

Hence, $\cos\theta=\dfrac{2}{3}$

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MPP-5 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Indefinite Integration

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. $\int x \sec^2 2x \, dx =$

(a) $\frac{x}{2}\tan 2x + \frac{1}{4}\ln \sin 2x + c$

(b) $\frac{x}{2}\tan 2x + \frac{1}{4}\ln \cos 2x + c$

(c) $\frac{x}{2}\sec 2x + \frac{1}{4}\ln \sin 2x + c$

(d) $\frac{x}{2}\sec 2x + \frac{1}{4}\ln \cos 2x + c$

2. $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}} =$

(a) $\frac{(x^4+1)^{\frac{1}{4}}}{x}+c$ (b) $-\frac{(x^4+1)^{\frac{1}{4}}}{x}+c$

(c) $\frac{\sqrt{x^4+1}}{x}+c$ (d) $-\frac{\sqrt{x^4+1}}{x}+c$

3. $\int \frac{\cos 2x \, dx}{(\cos x + \sin x)^2} =$

(a) $\frac{1}{2}\ln \sin\left(x+\frac{\pi}{4}\right)+c$ (b) $\frac{1}{2}\ln \cos\left(x+\frac{\pi}{4}\right)+c$

(c) $\ln \cos\left(x+\frac{\pi}{4}\right)+c$ (d) $\ln \cos\left(x-\frac{\pi}{4}\right)+c$

4. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx =$

(a) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (b) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$

(c) $\frac{\sqrt{2x^2-2x^2+1}}{x}+c$ (d) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

5. $\int e^{\ln\left(1+\frac{1}{x^2}\right)} \cdot \frac{x(x^2-1)}{x^4+1}dx =$

(a) $\ln\left(x^2+\frac{1}{x^2}\right)+c$ (b) $\ln\left(1+\frac{1}{x^2}\right)+c$

(c) $\frac{1}{2}\ln\left(x^2+\frac{1}{x^2}\right)+c$ (d) $\frac{1}{2}\ln\left(1+\frac{1}{x^2}\right)+c$

6. $\int \frac{dx}{(4+3x^2)\sqrt{3-4x^2}} =$

(a) $\frac{1}{5}\tan^{-1}\frac{2x}{\sqrt{3-4x^2}}+c$

(b) $\frac{1}{10}\tan^{-1}\left(\frac{5x}{2\sqrt{3-4x^2}}\right)+c$

(c) $\frac{1}{5}\tan^{-1}\frac{5x}{2\sqrt{3-4x^2}}+c$

(d) $\frac{1}{10}\tan^{-1}\frac{5x}{\sqrt{3-4x^2}}+c$

One or More Than One Option(s) Correct Type

7. If $\int \frac{3x+4}{x^3-2x-4}dx = \log|x-2| + K\log f(x) + c$, then

(a) $K = -(1/2)$
(b) $f(x) = x^2 + 2x + 2$
(c) $f(x) = |x^2 + 2x + 2|$
(d) $K = 1/4$

8. If the antiderivative of $\sin^{-1}\sqrt{\frac{x}{x+1}}$ is $x\sin^{-1}\sqrt{\frac{x}{x+1}} - \sqrt{x} + fog(x) + c$ then

(a) $f(x) = \sin^{-1}x$ (b) $g(x) = \sqrt{x+1}$
(c) $f(x) = \tan^{-1}x$ (d) $g(x) = \sqrt{x}$

9. If $\int \sqrt{\frac{\cos x - \cos^3 x}{1-\cos^3 x}}\,dx = f(x)$, then $f(x) =$

(a) $\frac{3}{2}\sin^{-1}(\cos^{3/2}x) + c$

(b) $-\cos^{-1}(\cos^{3/2}x) + c$

(c) $-\frac{2}{3}\sin^{-1}(\cos^{3/2}x) + c$

(d) $-\sin^{-1}(\cos^3 x) + c$

10. $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx$ is equal to

(a) $\sin x - 2\operatorname{cosec} x + 5\tan^{-1}(\cos x) + c$
(b) $\sin x - 2\operatorname{cosec} x - 6\tan^{-1}(\sin x) + c$
(c) $-[\operatorname{cosec} x + \cot x \cos x + 6\tan^{-1}(\sin x)] + c$
(d) $\sin x - 6\tan^{-1}(\sin x) + c$.

11. $\int \frac{dx}{\sin x + \sqrt{3}\cos x} =$

(a) $\frac{1}{2}\log\left|\sec\left(x-\frac{\pi}{6}\right) + \tan\left(x-\frac{\pi}{6}\right)\right| + c$

(b) $\frac{1}{2}\log\left|\operatorname{cosec}\left(x+\frac{\pi}{3}\right) - \cot\left(x+\frac{\pi}{3}\right)\right| + c$

(c) $-\frac{1}{2}\log\left|\operatorname{cosec}\left(x+\frac{\pi}{3}\right) + \cot\left(x+\frac{\pi}{3}\right)\right| + c$

(d) $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{6}\right) + c$.

12. If $\int \operatorname{cosec}(2x)dx = kf(g(x)) + c$, then

(a) range of $g(x) = (-\infty, \infty)$
(b) domain of $f(x) = R - \{0\}$
(c) $g'(x) = \sec^2 x$
(d) $f'(x) = 1/x$ for all $x \in (0, \infty)$.

13. $\int \frac{\log(x+1) - \log x}{x(x+1)}dx =$

(a) $\left[\log(x+1)^2 - \frac{1}{2}(\log x)\right]^2 + c$

(b) $-\frac{1}{2}[\log(x+1) - (\log x)]^2 + c$

(c) $c - \frac{1}{2}\left(\log\left(1+\frac{1}{x}\right)\right)^2$

(d) none of these

Comprehension Type

If the integrand is a rational function of x and fractional power of a linear fractional function of the form $\left(\frac{ax+b}{cx+d}\right)$ *i.e.*, $\int f\left(x, \left(\frac{ax+b}{cx+d}\right)^{m/n}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{r/s}\right)dx$.

In this form substitute $\frac{ax+b}{cx+d} = t^m$ where m is the LCM of the denominators of fractional powers of $\frac{ax+b}{cx+d}$.

On the basis of above information, answer the following questions:

14. The value of $\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}}$ is

(a) $2\lambda^{1/2}+3\lambda^{1/3}+6\lambda^{1/6}+6\ln|\lambda^{1/6}-1|+c$

(b) $2\lambda^{1/2}-3\lambda^{1/3}+6\lambda^{1/6}+6\ln|\lambda^{1/6}-1|+c$

(c) $2\lambda^{1/2}+3\lambda^{1/3}-6\lambda^{1/6}+6\ln|\lambda^{1/6}-1|+c$

(d) $2\lambda^{1/2}+3\lambda^{1/3}+6\lambda^{1/6}-6\ln|\lambda^{1/6}-1|+c$

where $\lambda = (1+x)$

15. The value of $\int \frac{1+x^{1/2}-x^{2/3}}{1+x^{1/3}}dx$ is

(a) $\frac{3}{4}x^{4/3}+\frac{6}{7}x^{7/6}+x+\frac{6}{5}x^{5/6}+2x^{1/2} - 6x^{1/6}+6\tan^{-1}(x^{1/6})+c$

(b) $-\frac{3}{4}x^{4/3}+\frac{6}{7}x^{7/6}+x+\frac{6}{5}x^{5/6}-2x^{1/2} + 6x^{1/6}-6\tan^{-1}(x^{1/6})+c$

(c) $\frac{3}{4}x^{4/3}-\frac{6}{7}x^{7/6}+x-\frac{6}{5}x^{5/6}+2x^{1/2} - 6x^{1/6}+6\tan^{-1}(x^{1/6})+c$

(d) $-\frac{3}{4}x^{4/3}+\frac{6}{7}x^{7/6}+x-\frac{6}{5}x^{5/6}+2x^{1/2}-6x^{1/6} + 6\tan^{-1}(x^{1/6})+c$

Matrix Match Type

16. Match the columns:

	Column I		Column II
(P)	$\int (e^{a\log x}+e^{x\log a})\,dx$	(1)	$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+c$
(Q)	$\int \frac{e^{\log\left(1+\frac{1}{x^2}\right)}}{x^2+\frac{1}{x^2}}dx$	(2)	$\tan^{-1}(\tan x+2)+c$
(R)	$\int \frac{dx}{\sin^2 x+4\sin x\cos x+5\cos^2 x}$	(3)	$\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$

	P	Q	R
(a)	2	1	3
(b)	2	3	1
(c)	3	1	2
(d)	1	2	3

Integer Answer Type

17. If $\int \frac{dx}{5+4\sin x} = \frac{2}{k}\left[\tan^{-1}\frac{(5\tan x/2)+4}{3}\right]$, then k is equal to

18. If $\int \frac{dx}{x^4(1+x^3)^2} = a\ln\left(\frac{1+x^3}{x^3}\right)+\frac{b}{x^3}+\frac{c}{1+x^3}+d$,

then $a+b+c=$

19. If $\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$

$=(Ax^2+Bx+C)\sqrt{x^2+4x+3}+\lambda\int\frac{dx}{\sqrt{x^2+4x+3}}$

then the value of $27A-6B-C$ is

20. If the integral

$\int \frac{5\tan x}{\tan x-2}dx = x+a\ln|\sin x-2\cos x|+k$, then a is equal to

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Keys are published in this issue. Search now! ☺

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 167

JEE MAIN

1. A triangle is inscribed in a rectangular hyperbola. Which one of the following points of the triangle lies on the hyperbola ?
   (a) centroid (b) incentre
   (c) circumcentre (d) orthocentre

2. $ABCD$ is an isosceles trapezium, $AB = 4$, $BC = DA = 3$, $CD = 2$. Circles of radii 2 are drawn with centres A and B and circles of radii 1 are drawn with centres C and D. The radius of the circle touching these four circles externally is
   (a) $\frac{3}{7}$ (b) $\frac{4}{7}$ (c) $\frac{5}{7}$ (d) $\frac{2}{7}$

3. Let $f(x)=\sum_{r=0}^{6} a_r x^r$, where $a_0 = 64$, $a_5 = -12$, $a_6 = 1$.
   If the roots of $f(x) = 0$ are positive, then $f''(1) =$
   (a) 12 (b) 24 (c) 30 (d) 60

4. Let $f(x)=\int_0^4 e^{|t-x|}dt$, $0 \le x \le 4$. If the range of $f(x)$ is $[a, b]$, then $\sqrt{b-a}$ is
   (a) e^2 (b) $e^2 + 1$ (c) $e^2 - 1$ (d) $e^2/2$

5. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 > 0$, $y_2 > 0$ be the ends of latus rectum of the ellipse $x^2 + 2y^2 = 2$. The equations of parabolas with latus rectum are
   (a) $x^2 - 2y = 1 - \sqrt{2}$ (b) $x^2 - 2y = \sqrt{2} + 2$
   (c) $x^2 + 2y = 2 - \sqrt{2}$ (d) None of these

JEE ADVANCED

6. Let $z_1 = 3 - 4i$, $z_2 = 2 + i$, $\alpha \ne 1$, $\alpha^6 = 1$.
   The sum $S=\sum_{r=0}^{5}\left|z_1+z_2\alpha^r\right|^2$ is divisible by
   (a) 2 (b) 3 (c) 5 (d) 7

COMPREHENSION

Let $y = f(x)$ such that $xy = x + y + 1$, $x \in R - \{1\}$ and $g(x) = x\,f(x)$

7. The minimum value of $g(x) =$
   (a) $3-\sqrt{2}$ (b) $3+\sqrt{2}$
   (c) $3-2\sqrt{2}$ (d) $3+2\sqrt{2}$

8. The area bounded by the curve $y = g(x)$ and the x-axis is
   (a) $3/2 + \ln 4$ (b) $3/2 - \ln 4$
   (c) $1/2 + \ln 4$ (d) $\ln 4 - 1$

INTEGER MATCH

9. Let PQ and PR are the focal chords of the ellipse $x^2 + 2y^2 = 2$. If $p=\left(1,\frac{1}{\sqrt{2}}\right)$ and $QR = d$, then $[d^2]$ is

MATRIX MATCH

10.

	Column I		Column II
(a)	$\lim_{x\to\infty}\int_0^x x e^{t^2-x^2}dt =$	(p)	$\frac{1}{3}$
(b)	Let $f(x)$ be differentiable such that the function $f(x)=x^2+\int_0^x e^{-t} f\,\vert x-t\vert\,dt$, then $f(1) =$	(q)	$\frac{1}{2}$
(c)	If $f(x)=\int_0^1\vert x-t\vert\,dt$, $x\in[0,1]$ then $\int_0^1 f(x)dx =$	(r)	1
(d)	$\int_0^{\pi/2}\frac{1+2\cos x}{(2+\cos x)^2}\cdot dx =$	(s)	$\frac{2}{3}$
		(t)	$\frac{4}{3}$

See Solution Set of Maths Musing 166 on page no. 83

WB JEE MOCK TEST PAPER

Series-4

The entire syllabus of Mathematics of WB-JEE is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given below.

Unit No.	Topic	Syllabus In Details
UNIT NO. 4	Mathematical Induction	Principle of mathematical induction and its simple application
	Binomial Theorem & it's Simple Applications	Binomial Theorem of positive integral index, general term and middle term, properties of Binomial coefficients and simple applications
	Sequences and Series	Definition of series and sequence. Arithmetic, geometric and harmonic progressions. Arithmetic, geometric & harmonic means between two given numbers, relation between A.M., G.M. and H.M., Sum upto n terms of special series: $\sum n$, $\sum n^2$, $\sum n^3$. Arithmetic-Geometric progression.
	Co-ordinate geometry-2D	Conic Sections: Sections of cones, equation conic sections(parabola, ellipse and hyperbola) in standard forms, condition for $y = mx + c$ to be a tangent and point(s) of tangency

Time : 1 hr 15 min **Full marks : 45**

CATEGORY-I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. The middle term of an arithmetic progression consisting of n number of terms is m, the sum of the terms of the series will be
(a) $m + n$ (b) mn (c) $m - n$ (d) m/n

2. The 5th term of a G.P. is 2, then the product of the first 9 terms is
(a) 256 (b) 128
(c) 512 (d) none of these

3. If $\frac{1}{ab+ac}, \frac{1}{bc+ba}, \frac{1}{ca+cb}$ are in H.P., then a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

4. If a, x, b are in A.P. and a, y, z, b are in G.P., then the value of $\frac{y^3+z^3}{xyz}$ is
(a) 1 (b) 2
(c) 1/2 (d) none of these

5. If x be the arithmetic mean of two positive numbers and y and z are two geometric means between them, then the harmonic mean of the numbers will be
(a) $\frac{x}{yz}$ (b) $\frac{yz}{x}$ (c) $\frac{2yz}{x}$ (d) $\frac{2x}{yz}$

6. If $S_n = nP + \frac{n(n-1)}{2}Q$, where S_n denotes the sum of the first n terms of an A.P., then the common difference is
(a) $P + Q$ (b) $2P + 3Q$
(c) $2Q$ (d) Q

7. If a, b, c, d are four numbers such that the first three are in A.P., while the last three are in H.P., then
(a) $ab = cd$ (b) $ac = bd$
(c) $ad = bc$ (d) none of these

8. The sum of integers from 1 to 100 which are divisible by 2 or 5, is
(a) 3000 (b) 3010 (c) 3150 (d) 3050

9. If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y, z respectively then $x^{q-r} \times y^{r-p} \times z^{p-q} =$
(a) 0 (b) 1
(c) xyz (d) none of these

By : Sankar Ghosh, S.G.M.C, Kolkata, Ph: 09831244397.

10. The sum of n terms of the series $\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}+ \ldots$ is

(a) $n+\frac{1}{3}(3^n-1)$ (b) $n-\frac{1}{3}(3^n-1)$

(c) $n-\frac{1}{2}(3^{-n}-1)$ (d) $n-\frac{1}{2}(1-3^{-n})$

11. If 5^{40} is divided by 11, then remainder is

(a) 2 (b) 3 (c) 5 (d) 1

12. The number of terms in the expansion of $(a + b + c)^n$, where $n \in N$ is

(a) $\frac{(n+1)(n+2)}{2}$ (b) $n + 1$

(c) $n + 2$ (d) $(n + 1)n/2$

13. The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is

(a) 1 (b) 5/12

(c) ${}^{10}C_1$ (d) none of these

14. If the coefficient of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55 (c) 45 (d) 15

15. Given the integers $r > 1$, $n > 2$ and coefficients of $(3r)^{\text{th}}$ and $(r + 2)^{\text{th}}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then

(a) $n = 2r$ (b) $n = 3r$

(c) $n = 2r + 1$ (d) none of these

16. The digit at unit's place in the number $17^{2017} + 11^{2017} - 7^{2017}$ is

(a) 0 (b) 1 (c) 2 (d) 3

17. If $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$ then

(a) $\text{Re}(z) = 0$ (b) $\text{Im}(z) = 0$

(c) $\text{Re}(z) > 0, \text{Im}(z) > 0$

(d) $\text{Re}(z) > 0, \text{Im}(z) < 0$

18. The number of integral terms in the expansion of $(2\sqrt{5}+\sqrt[6]{7})^{642}$ is

(a) 105 (b) 107 (c) 321 (d) 108

19. If the coefficient of the second, third and fourth terms in the expansion of $(1 + x)^n$ are in A.P., then n equal to

(a) 2 (b) 7

(c) 9 (d) none of these

20. The focus of the parabola whose vertex is (3, 2) and whose directrix is $x - y + 1 = 0$ is

(a) (4, 1) (b) (1, –1)

(c) (8, 7) (d) (–4, 1)

21. If the slope of a chord OP of the parabola $y^2 = 4ax$ (where O is the vertex) is $\tan\alpha$, then the length of the chord is

(a) $4a \tan\alpha \sec\alpha$ (b) $4a \tan\alpha \cos\alpha$

(c) $4a \cot\alpha \operatorname{cosec}\alpha$ (d) $4a \sin\alpha \tan\alpha$

22. M is the foot of the perpendicular from a point P on $y^2 = 4ax$ to its directrix. If S is the focus and the ΔSPM is equilateral, then its area is equal to

(a) $4\sqrt{3}a^2$ (b) $2\sqrt{3}a^2$

(c) $8\sqrt{3}a^2$ (d) $16a^2$

23. Two conics $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $x^2=-\frac{a}{b}y$ intersect if

(a) $0<b<\frac{1}{2}$ (b) $0<a<\frac{1}{2}$

(c) $a^2 < b^2$ (d) $a^2 > b^2$

24. An ellipse has eccentricity 1/2 and one focus at $S\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent, (nearer to S) to the circle $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$. The equation of the ellipse in standard form is

(a) $9\left(x-\frac{1}{3}\right)^2+12(y-1)^2=1$

(b) $12\left(x-\frac{1}{3}\right)^2+9(y-1)^2=1$

(c) $\frac{\left(x-\frac{1}{2}\right)^2}{12}+\frac{(y-1)^2}{9}=1$

(d) $3\left(x+\frac{1}{2}\right)^2+4(y-1)^2=1$

25. The eccentricity of the hyperbola $x^2 - 3y^2 + 1 = 0$ is

(a) 3 (b) 2 (c) 3/2 (d) 4/3

26. If C is the centre and A, B are two points on the conic $4x^2 + 9y^2 - 8x - 36y + 4 = 0$ such that $\angle ACB=\frac{\pi}{2}$, then $(CA)^{-2} + (CB)^{-2}$ is equal to

(a) $\frac{13}{36}$ (b) $\frac{36}{13}$ (c) $\frac{16}{33}$ (d) $\frac{33}{16}$

27. If e_1 is the eccentricity of the conic $9x^2 + 4y^2 = 36$ and e_2 is the eccentricity of the conic $9x^2 - 4y^2 = 36$, then

(a) $e_1^2 + e_2^2 = 2$ (b) $3 < e_1^2 + e_2^2 < 4$

(c) $e_1^2 + e_2^2 > 4$ (d) none of these

28. A tangent having slope –4/3 to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major and minor axes at A and B. If O is the origin, then the area of the ΔOAB is (in sq. units)

(a) 48 (b) 9 (c) 24 (d) 16

29. Tangents at any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut the axes at A and B respectively. If the rectangle $OAPB$ (where O is origin) is completed then the locus of point P is given by

(a) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$ (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$

(c) $\frac{a^2}{y^2}-\frac{b^2}{x^2}=1$ (d) none of these

CATEGORY-II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

30. Let $p \geq 3$ is a positive integer and α and β are two roots of the equation $x^2 - (p + 1)x + 1 = 0$, then $\alpha^n + \beta^n$ (where $n \in N$) =

(a) 0 (b) an integer

(c) a proper fraction (d) none of these

31. The middle term in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is

(a) $\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{n!}\cdot x^n$

(b) $\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}{n!}\cdot 2^n$

(c) $\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{(n+1)!}\cdot 2^n$

(d) $\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{n!}\cdot 2^n$

32. The sum to n-terms of the series

$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots\ldots$

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$

(c) $n + 2^{-n} - 1$ (d) $2^n - 1$

33. The value of c for which the line $y = 3x + c$ touches the ellipse $16x^2 + y^2 = 16$ is

(a) 5 (b) 1 (c) 4 (d) 3

CATEGORY-III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks.

2×(no. of correct response/total no. of correct options)

34. Let $n \in N$ then $x(x^{n-1} - na^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$ when

(a) $n = 2$ (b) $n = 3$

(c) $\forall\ n \in N$ (d) none of these

35. The 6^{th} term of the expansion $\left[\sqrt{2^{\log_{10}(10-3^x)}}+\sqrt[5]{2^{(x-2)\log_{10}3}}\right]^m$ is 21 and the coefficient of 2^{nd}, 3^{rd} and 4^{th} terms of it are respectively 1^{st}, 3^{rd} and 5^{th} term of an A.P. Find x.

(a) 0 (b) 1 (c) 2 (d) 3

36. A chord PP' of a parabola cuts the axis of the parabola at O. The feet of the perpendiculars from P and P' on the axis are M and M' respectively. If V is the vertex then VM, VO, VM' are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

37. A focus of the hyperbola $25x^2 - 36y^2 = 225$ is

(a) $(\sqrt{61}, 0)$ (b) $\left(\frac{1}{2}\sqrt{61}, 0\right)$

(c) $(-\sqrt{61},\ 0)$ (d) $\left(-\frac{1}{2}\sqrt{61}, 0\right)$

SOLUTIONS

1. (b): Given that, the middle term of an A.P., consisting of n terms is m. Since there is only one middle term, therefore, n is odd. Let $n = 2p + 1$

Thus $t_{p+1} = m \Rightarrow a + pd = m$ [where a and d are respectively the first term and the common difference]

Now, $S_n = \frac{2p+1}{2}\{2a+(2p+1-1)d\}$

$= (2p + 1)(a + pd) = m(2p + 1) = mn$

$\therefore\ S_n = mn$

2. (c) : Let the first term be a and the common ratio be r.

$\therefore\ t_5 = 2 \Rightarrow ar^4 = 2$

Now $a\cdot ar\cdot ar^2\cdot ar^3\ldots\cdot ar^8 = a^9 r^{1+2+3+\ldots+8}$

$=(ar^4)^9 = 2^9 = 512$

3. (c) : Given that

$$\frac{1}{ab+ac}, \frac{1}{bc+ba}, \frac{1}{ca+cb} \in \text{H.P.}$$

$\Rightarrow$ $ab + ac, bc + ba, ca + cb \in$ A.P.

$\Rightarrow$ $-bc, -ca, -ab \in$ A.P. $\Rightarrow$ $bc, ca, ab \in$ A.P.

$$\Rightarrow \frac{bc}{abc}, \frac{ca}{abc}, \frac{ab}{abc} \in \text{A.P.} \Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \in \text{A.P.}$$

$\therefore$ $a, b, c \in$ H.P.

4. (b) : Given that $a, x, b \in$ A.P. $\therefore$ $x = \frac{a+b}{2}$ ...(i)

Again $a, y, z, b \in$ G.P. $\therefore$ $y^2 = az$ and $z^2 = by$

$$\Rightarrow a = \frac{y^2}{z} \text{ and } b = \frac{z^2}{y}$$

Now from (i), we get

$$\frac{y^2}{z} + \frac{z^2}{y} = 2x \Rightarrow \frac{y^3+z^3}{yz} = 2x \Rightarrow \frac{y^3+z^3}{xyz} = 2$$

5. (b) : Let the two given numbers be a and b

$$\therefore \quad x = \frac{a+b}{2}$$

And $y^2 = az$ and $z^2 = by \Rightarrow a = \frac{y^2}{z}$ and $b = \frac{z^2}{y}$

Now, H.M. of a and b is

$$H = \frac{2ab}{a+b} = \frac{2 \times \frac{y^2}{z} \times \frac{z^2}{y}}{2x} = \frac{yz}{x}$$

6. (d) : Here $S_n = nP + \frac{n(n-1)Q}{2}$

$$\Rightarrow S_n = \left(\frac{1}{2}Q\right)n^2 - \left(\frac{1}{2}Q - P\right)n$$

$\therefore$ Common difference $= \left(\frac{1}{2}Q\right) \times 2 = Q$

7. (c) : Given that $a, b, c \in$ A.P. $\therefore$ $2b = a + c$

Again $b, c, d \in$H.P. $\therefore$ $c = \frac{2bd}{b+d}$

$\Rightarrow$ $2b - a = \frac{2bd}{b+d} \Rightarrow (2b - a)(b + d) = 2bd$

$\Rightarrow$ $2b^2 - ab = ad$ $\Rightarrow$ $b(2b - a) = ad$

$\Rightarrow$ $b(a + c - a) = ad$ $\Rightarrow$ $bc = ad$

8. (d) : Sum of the integers from 1 to 100 which are divisible by 2 or 5 is

$(2 + 4 + 6 + ... + 100) + (5 + 10 + 15 + ... + 100)$
$- (10 + 20 + 30 + ... + 100)$

$$= \frac{50}{2}(2+100) + \frac{20}{2}(5+100) - \frac{10}{2}(10+100) = 3050$$

9. (b) : Let the first term and common ratio of the G.P. respectively be m and n.

Then, $t_p = x$ $\Rightarrow$ $mn^{p-1} = x$

$t_q = y$ $\Rightarrow$ $mn^{q-1} = y$

$t_r = z$ $\Rightarrow$ $mn^{r-1} = z$

Now, $x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$

$$= (mn^{p-1})^{q-r} \cdot (mn^{q-1})^{r-p} \cdot (mn^{r-1})^{p-q}$$

$$= m^{q-r+r-p+p-q} \cdot n^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$

$$= m^0 n^0 = 1$$

10. (d) : Let $S_n = \frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + ...$

$$= \frac{3-1}{3} + \frac{3^2-1}{9} + \frac{3^3-1}{27} + \frac{3^4-1}{81} + ...$$

$$= (1+1+... \text{ to } n \text{ terms}) - \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + ...\text{to } n \text{ terms}\right)$$

$$= n - \frac{\frac{1}{3}\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}} = n - \frac{1}{2}(1-3^{-n})$$

11. (d) : Here, $5^{40} = (5^2)^{20} = (22 + 3)^{20}$, so the remainder is same as 3^{20}.

Now $3^{20} = (3^2)^{10} = (11 - 2)^{10}$, so the remainder is same as 2^{10}.

But $2^{10} = 1024 = 11 \times 93 + 1$.

So, the remainder is 1.

12. (d) : $(a + b + c)^n = a^n + {}^nC_1 a^{n-1}(b + c)$
$+ {}^nC_2\, a^{n-2}(b + c)^2 + {}^nC_3\, a^{n-3}(b + c)^3 + ...$
$+ {}^nC_n(b + c)^n$

Further expanding each term of R.H.S., first term gives one term, second term gives two terms, third term gives three terms and so on.

$\therefore$ Total number of terms $= 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

13. (d) : Let t_{r+1}th term is independent of x.

$$\therefore \quad t_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\sqrt{\frac{3}{2x^2}}\right)^r$$

$$= {}^{10}C_r\, x^{\frac{1}{2}(10-3r)} \cdot 3^{r-5} \cdot \frac{1}{2^{r/2}}$$

For independent of x terms, $\frac{10-3r}{2} = 0 \Rightarrow r = \frac{10}{3}$

This is not an integer. Therefore, there will be no constant term.

14. (b)

15. (a) : The coefficient of $(3r)^{\text{th}}$ and $(r + 2)^{\text{th}}$ terms will be ${}^{2n}C_{3r-1}$ and ${}^{2n}C_{r+1}$ respectively.

By the problem, ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow\ 3r - 1 + r + 1 = 2n$

$\Rightarrow\ 4r = 2n \quad \Rightarrow \quad n = 2r$

16. (b) : $17^{2017} + 11^{2017} - 7^{2017}$

$= (7 + 10)^{2017} + (1 + 10)^{2017} - 7^{2017}$

$$=[7^{2017} + {}^{2017}C_1 \cdot 7^{2016} \cdot 10 + {}^{2017}C_2 \cdot 7^{2015} \cdot 10^2 + \ldots$$
$$\ldots + {}^{2017}C_{2017} \cdot 10^{2017}]$$
$$+[1 + {}^{2017}C_1 \cdot 10 + {}^{2017}C_2 \cdot 10^2 + \ldots + 10^{2017}] - 7^{2017}$$
$$=[{}^{2017}C_1 \cdot 7^{2016} \cdot 10 + {}^{2017}C_2 \cdot 7^{2015} \cdot 10^2 + \ldots$$
$$\ldots + {}^{2017}C_{2017} \cdot 10^{2017}]$$
$$+ [{}^{2017}C_1 \cdot 10 + {}^{2017}C_2 \cdot 10^2 + \ldots + 10^{2017}] + 1$$

= a multiple of 10 + 1

Thus unit's place digit is 1

17. (b)

18. (d) : The general term is.

$$t_{r+1} = {}^{642}C_r (2\sqrt{5})^{642-r} (\sqrt[6]{7})^r = {}^{642}C_r\, 2^{642-r} 5^{321-\frac{r}{2}} \cdot 7^{\frac{r}{6}}$$

The above term will be integral if and only if r is a multiple of 6.

Thus r takes the values 0, 6, 12, 18, …, 642.

Therefore the number of integral terms in the given expansion is 108.

19. (b) : According to the given problem we have ${}^nC_1$, ${}^nC_2$, ${}^nC_3$ are in A.P. $\Rightarrow {}^nC_1 + {}^nC_3 = 2\,{}^nC_2$

$$\Rightarrow\ n + \frac{n(n-1)(n-2)}{6} = 2 \cdot \frac{n(n-1)}{2}$$

$$\Rightarrow\ 1 + \frac{(n-1)(n-2)}{6} = n - 1$$

$\Rightarrow\ 6 + (n - 1)(n - 2) = 6n - 6$

$\Rightarrow\ n^2 - 3n + 2 = 6n - 12$

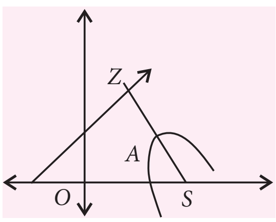
$\Rightarrow\ n^2 - 9n + 14 = 0 \quad \Rightarrow \quad n = 7, 2$

But $n = 2$ is rejected as ${}^nC_3$ is not possible.

20. (a) : The equation of the axis of the parabola is

$y - 2 = -1(x - 3) \quad \Rightarrow \quad x + y - 5 = 0$

Hence the point of intersection of axis and directrix of the parabola is $Z = (2, 3)$



A (vertex) is the mid-point of ZS (S is the focus)

$\therefore\ S = (4, 1)$

21. (c) : Slope of $OP = \dfrac{2at}{at^2} = \dfrac{2}{t} = \tan\alpha$ (given).

Length of the chord $OP = \sqrt{a^2t^4 + 4a^2t^2} = at\sqrt{t^2 + 4}$

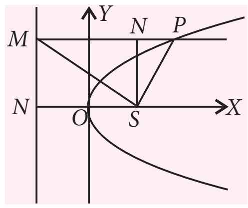
$$= \frac{2a}{\tan\alpha}\sqrt{\frac{4}{\tan^2\alpha} + 4} = 4a\cot\alpha\,\mathrm{cosec}\,\alpha$$

22. (a) : From the diagram, we have

From ΔNSM, $\dfrac{MN}{NS} = \tan 30°$

$$\Rightarrow\ \frac{2a}{2at} = \frac{1}{\sqrt{3}}$$

$\therefore\ t = \sqrt{3} \Rightarrow MP = a + at^2 = 4a$



Area of $\Delta SPM = \dfrac{1}{2} MP \times SN$

$$= \frac{1}{2}(4a)(2at) = 4a^2t = 4\sqrt{3}a^2$$

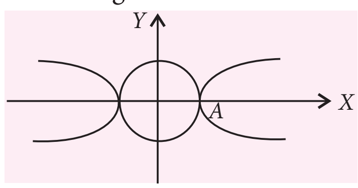
23. (b) : Eliminating x, we have $\dfrac{y^2}{b^2} + \dfrac{y}{ab} + 1 = 0$

The given curves will intersect each other if

$$\frac{1}{a^2b^2} - \frac{4}{b^2} > 0 \Rightarrow \frac{1}{a^2} > 4 \Rightarrow a^2 < \frac{1}{4} \Rightarrow a < \frac{1}{2}$$

Hence the conics intersect if $0 < a < \dfrac{1}{2}$.

24. (a) : For the circle $x^2 + y^2 = 1$ and rectangular hyperbola $x^2 - y^2 = 1$, one common tangent is evidently $x = 1$, the other being $x = -1$.



We require in standard form the ellipse with focus at $S\left(\dfrac{1}{2}, 1\right)$ and directrix $x = 1$ which is

$$\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \left(\frac{1}{2}\right)^2 (1-x)^2$$

$$\Rightarrow\ \frac{3x^2}{4} - \frac{x}{2} + (y-1)^2 = 0$$

$$\Rightarrow\ \frac{3}{4}\left(x - \frac{1}{3}\right)^2 + (y-1)^2 = \frac{1}{12}$$

$$\Rightarrow\ 9\left(x - \frac{1}{3}\right)^2 + 12(y-1)^2 = 1$$

25. (b) : The given equation of the conic is

$$x^2-3y^2+1=0 \;\Rightarrow\; \frac{y^2}{\frac{1}{3}}-\frac{x^2}{1}=1,$$

which is the conjugate hyperbola .

$\therefore\quad 1=\frac{1}{3}(e^2-1)$, which gives $e = 2$.

26. (a) : The equation can be written as

$4(x - 1)^2 + 9(y - 2)^2 = 36$

which is an ellipse centred at (1, 2).

If CA makes an angle θ with the major axis, then

$$A\equiv\left[1+CA\cos\theta, 2+CA\sin\theta\right],$$

$$B\equiv\left[1+CB\cos\left(\frac{\pi}{2}+\theta\right), 2+CB\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

As A and B are two points on the conic

$\therefore\quad CA^2(4\cos^2\theta + 9\sin^2\theta) = 36$ and

$CB^2(4\sin^2\theta + 9\cos^2\theta) = 36$

giving $(CA)^{-2}+(CB)^{-2}=\frac{13}{36}$

27. (b) : The given equations in standard form are

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \text{ and } \frac{x^2}{4}-\frac{y^2}{9}=1.$$

Now, $e_1^2=\frac{9-4}{9}=\frac{5}{9},\quad e_2^2=\frac{4+9}{4}=\frac{13}{4}$

and $e_1^2+e_2^2=\frac{137}{36}>3$ but <4

28. (c) : Any point on the ellipse $\frac{x^2}{(3\sqrt{2})^2}+\frac{y^2}{(4\sqrt{2})^2}=1$ can be taken as $(3\sqrt{2}\cos\theta,\ 4\sqrt{2}\sin\theta)$ and slope of the tangent

$$=-\frac{b^2x}{a^2y}=-\frac{32(3\sqrt{2}\cos\theta)}{18(4\sqrt{2}\sin\theta)}=-\frac{4}{3}\cot\theta \qquad \text{... (i)}$$

Given slope of the tangent $=-\frac{4}{3}$ ... (ii)

From equation (i) and (ii), $\cot\theta=1 \;\Rightarrow\; \theta=\frac{\pi}{4}$

Hence equation of tangent is

$$\frac{x\cdot\frac{1}{\sqrt{2}}}{3\sqrt{2}}+\frac{y\cdot\frac{1}{\sqrt{2}}}{4\sqrt{2}}=1 \;\Rightarrow\; \frac{x}{6}+\frac{y}{8}=1$$

Hence $A \equiv (6, 0)$, $B = (0, 8)$

Area of $\Delta OAB=\frac{1}{2}\times 6\times 8=24$ sq.units.

29. (a) : Equation of tangent at the point θ is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1 \Rightarrow A(a\cos\theta,\ 0), B(0, -b\cot\theta)$$

Let P be (h, k) $\Rightarrow$ $h = a\cos\theta$, $k = -b\cot\theta$

$$\Rightarrow \frac{k}{h}=-\frac{b}{a\sin\theta} \;\Rightarrow\; \sin\theta=-\frac{bh}{ak}\Rightarrow \frac{b^2h^2}{a^2k^2}+\frac{h^2}{a^2}=1$$

$$\Rightarrow \frac{b^2}{k^2}+1=\frac{a^2}{h^2} \;\Rightarrow\; \frac{a^2}{h^2}-\frac{b^2}{k^2}=1.$$

Thus the locus of P is $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1.$

30. (b) : Given that α and β are the roots of the equation $x^2 - (p + 1)x + 1 = 0$

$\therefore\quad \alpha + \beta = p + 1$ and $\alpha\beta = 1$

If $n = 1$, then $\alpha^n + \beta^n = \alpha + \beta = p + 1$, which is an integer ($\because p \ge 3$ is a positive integer)

If $n = 2$, then $\alpha^n + \beta^n = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (p + 1)^2 - 2 \cdot 1 = p^2 + 2p - 1$

which is an integer ($\because p \ge 3$ is a positive integer)

Let $p(n) : \alpha^n + \beta^n$ = an integer

$\therefore\quad p(1)$ and $p(2)$ are true.

So, it can be prove by the principle of mathematical induction that the statement $p(n)$ is true.

31. (d) : The required middle term is

$$t_{n+1}={}^{2n}C_n\cdot x^{2n-n}\cdot\left(\frac{1}{x}\right)^n={}^{2n}C_n x^n\cdot\frac{1}{x^n}$$

$$={}^{2n}C_n=\frac{\lfloor 2n}{\lfloor n\lfloor n}=\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\ldots\ldots\cdot(2n-1)2n}{\lfloor n\lfloor n}$$

$$=\frac{(1\cdot 3\cdot 5\ldots.(2n-1))(2\cdot 4\cdot 6\cdot\ldots\cdot 2n)}{\lfloor n\lfloor n}$$

$$=\frac{(1\cdot 3\cdot 5\ldots.(2n-1))2^n\lfloor n}{\lfloor n\lfloor n}=\frac{(1\cdot 3\cdot 5\ldots.(2n-1))}{\lfloor n}\cdot 2^n$$

32. (c) : Let the sum of the first n terms of the series be S and its n^{th} term be t_n.

$$\therefore\quad S=\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots..+t_n$$

$$S=\qquad\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\ldots\ldots+t_{n-1}+t_n$$

(Subtracting) $0=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ to n terms $-t_n$

$$\Rightarrow \quad t_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\text{..... to } n \text{ terms} = 1-\frac{1}{2^n}$$

$$\text{Thus, } S = \left(1-\frac{1}{2}\right)+\left(1-\frac{1}{2^2}\right)+\left(1-\frac{1}{2^3}\right)+.....+\left(1-\frac{1}{2^n}\right)$$

$$= n-\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+....+\frac{1}{2^n}\right) = n-\left(1-\frac{1}{2^n}\right) = n-1+2^{-n}.$$

33. (a) : The given equation of the ellipse is

$$16x^2 + y^2 = 16 \Rightarrow \frac{x^2}{1}+\frac{y^2}{16}=1$$

The line which touches the above ellipse is $y = 3x + c$
Here, $a = 1$, $b = 4$, $m = 3$

$$\therefore \quad c=\sqrt{a^2m^2+b^2}=\sqrt{1^2\cdot 3^2+4^2}=5$$

34. (a, b, c) : If $n = 1$, then

$$x(x^{n-1}-na^{n-1})+a^n(n-1)=x(1-1\cdot a^0)+(1-1)\cdot a$$
$$= x \cdot 0 + 0 \cdot a = 0$$

which is divisible by $(x - a)^2$
If $n = 2$, then

$$x(x^{n-1}-na^{n-1})+a^n(n-1)=x(x-2a)+a^2(2-1)$$
$$= x^2 - 2xa + a^2 = (x - a)^2$$

which is divisible by $(x - a)^2$
If $n = 3$, then

$$x(x^{n-1}-na^{n-1})+a^n(n-1)=x(x^2-3a^2)+2a^3$$
$$= x^3 - 3a^2x + 2a^3 = x^3 - a^2x - 2a^2x + 2a^3$$
$$= x(x^2 - a^2) - 2a^2(x - a) = (x - a)(x^2 + ax - 2a^2)$$
$$= (x - a)(x - a)(x + 2a) = (x - a)^2(x + 2a)$$

which is divisible by $(x - a)^2$
$\therefore$ For any $n \in N$, $x(x^{n-1} - na^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$.

35. (a, c) : The coefficients of 2nd, 3rd and 4th terms of the given expansion are respectively ${}^mC_1$, ${}^mC_2$, ${}^mC_3$. Also these are respectively the 1st, 3rd and 5th term of an A.P. Again the 1st, 3rd and 5th terms of an A.P. are also in A.P.
Therefore, $2({}^mC_2) = {}^mC_1 + {}^mC_3 \Rightarrow m = 2, 7$
$\because$ The 6th term of the given expansion is 21,
$\therefore$ $m \neq 2$, so, $m = 7$
Now the 6th term of the expansion is

$$t_6 = {}^7C_5\left\{\sqrt{2^{\log_{10}(10-3^x)}}\right\}^{7-5}\cdot\left\{\sqrt[5]{2^{(x-2)\log_{10}3}}\right\}^5$$

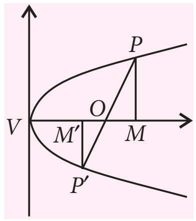
$$\Rightarrow \quad {}^7C_5\cdot 2^{\log_{10}(10-3^x)}\cdot 2^{(x-2)\log_{10}3}=21$$

$$\Rightarrow \quad 2^{\log_{10}\{(10-3^x)\cdot 3^{x-2}\}}=1=2^0$$

$$\Rightarrow \quad \{(10 - 3^x)\cdot 3^{x-2}\} = 1$$

Solving we get $x = 0, 2$

36. (b) : Let the parabola be



$y^2 = 4ax$
and $P(at_1^2, 2at_1)$, $P'(at_2^2, 2at_2)$
$\therefore \quad VM = at_1^2,\ VM' = at_2^2$
and $VO = k$

$$\text{where } \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ k & 0 & 1 \end{vmatrix} = 0 \ (\because P, P', O \text{ are collinear})$$

$\therefore \quad VM\cdot VM' = (at_1t_2)^2 = k^2 = VO^2$

37. (b, d) : The given equation of the hyperbola is

$$25x^2-36y^2=225 \Rightarrow \frac{x^2}{9}-\frac{y^2}{\left(\frac{5}{2}\right)^2}=1$$

$$\therefore \quad a^2=9 \text{ and } b^2=\left(\frac{5}{2}\right)^2 \Rightarrow e=\sqrt{1+\frac{25}{36}}=\frac{\sqrt{61}}{6}$$

$$\therefore \quad \text{Focus is } (\pm ae, 0)=\left(\pm\frac{\sqrt{61}}{2}, 0\right)$$

JEE WORK OUTS

PAPER-I

SECTION-1

This section contains eight questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. How many solutions are there for the equation, $|\cos x - \sin x| = 2\cos x$ in the interval $[0, 2\pi]$?

2. The number of solutions of the system $\log_2 x \log_y 2 + 1 = 0$ and $\sin x \cos y = 1 - \cos x \sin y$, which satisfy the condition $x + y < 8$ is

3. $\tan 100° + \tan 125° + \tan 100° \tan 125° =$

4. If $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then $\left(\dfrac{\tan\alpha}{\tan\beta}\right)^2$ is equal to

5. If $A + B + C = 180°$, $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = k \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$, then the value of k is equal to

6. The expression $\tan 55° \cdot \tan 65° \cdot \tan 75°$ simplifies to $\cot x°$, where $x \in (0, 90)$ then x equals

7. The maximum value of $y = 2\sin^2 x - 3\sin x + 1\ \forall\ x \in R$ is

8. Let x and y be positive real numbers and θ be an angle such that $\theta \neq \dfrac{n\pi}{2}$, $n \in I$. Suppose $\dfrac{\sin\theta}{x} = \dfrac{\cos\theta}{y}$ and $\dfrac{\cos^4\theta}{x^4} + \dfrac{\sin^4\theta}{y^4} = \dfrac{97\sin 2\theta}{x^3 y + y^3 x}$, then the value of $\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$ is equal to

SECTION-2

This section contains ten questions. Each question has four options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubbles corresponding to all the correct options in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened. –2 In all other cases.

9. The vectors $\vec a, \vec b, \vec c$ are of the same length and the angle between any two of them is the same. If $\vec a = \hat i + \hat j$, $\vec b = \hat j + \hat k$ then $\vec c$ is

   (a) $-\dfrac{1}{3}(\hat i - 4\hat j + \hat k)$ (b) $\hat i + \hat k$

   (c) $\dfrac{1}{3}(-\hat i + 5\hat j + \hat k)$ (d) $(\hat i + 2\hat j + 3\hat k)$

10. The lines whose vector equations are $\vec r = 2\hat i - 3\hat j + 7\hat k + \lambda(2\hat i + p\hat j + 5\hat k)$ and $\vec r = \hat i + 2\hat j + 3\hat k + \mu(3\hat i - p\hat j + p\hat k)$ are perpendicular for all values of λ and μ if

    (a) $p = -6$ (b) $p = -1$

    (c) $p = 1$ (d) $p = 6$

11. The vectors $\vec a = x\hat i - 2\hat j + 5\hat k$ and $\vec b = \hat i + y\hat j - z\hat k$ are collinear if

    (a) $x = 1, y = -2, z = -5$

    (b) $x = 1/2, y = -4, z = -10$

    (c) $x = -1/2, y = 4, z = 10$

    (d) $x = -1, y = 2, z = 5$

12. Vector equation of the line $6x - 2 = 3y + 1 = 2z - 2$ is

    (a) $\vec r = \hat i - \hat j + 3\hat k + \lambda(\hat i + 2\hat j + 3\hat k)$

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(b) $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}\right)$

(c) $\vec{r}=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}+\lambda(\hat{i}+2\hat{j}+3\hat{k})$

(d) $\vec{r}=-2\hat{i}+\hat{j}-2\hat{k}+\lambda(6\hat{i}+3\hat{j}+2\hat{k})$

where, λ being a parameter.

13. The equations of the line of shortest distance between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ are

(a) $3(x - 21) = 3y + 92 = 3z - 32$

(b) $\frac{x-(62/3)}{1/3}=\frac{y+31}{1/3}=\frac{z-(31/3)}{1/3}$

(c) $\frac{x-21}{1/3}=\frac{y+(92/3)}{1/3}=\frac{z-(32/3)}{1/3}$

(d) $\frac{x-2}{1/3}=\frac{y+3}{1/3}=\frac{z-1}{1/3}$

14. If Δ denotes the area of the triangle ABC, then Δ is equal to

(a) $ab\sin C$ (b) $\frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)}$

(c) $\frac{1}{2}(a+b+c)r$ (d) $\frac{1}{2}(a+b-c)r$

15. $\sqrt{\cos 2x}+\sqrt{1+\sin 2x}=2\sqrt{\sin x+\cos x}$ if

(a) $x=n\pi-\frac{\pi}{3}$ (b) $x = 2n\pi$

(c) $x=n\pi-\frac{\pi}{4}$ (d) $x=2n\pi\pm\cos^{-1}\left(-\frac{1}{5}\right)$

16. If A and B are angles such that $A+B=\frac{\pi}{3}$ and $y = \tan A \tan B$, then y can be equal to

(a) 0 (b) 2 (c) 4 (d) 5

17. The equation $\sin^4 x + \cos^4 x = a$ has a solution for

(a) all values of a (b) $a = 1$

(c) $a = 1/2$ (d) $a = 3/4$

18. If $\theta=\tan^{-1}(2\tan^2\theta)-\tan^{-1}\left(\left(\frac{1}{3}\right)\tan\theta\right)$, then θ =

(a) 0 (b) $\pi/4$

(c) $\tan^{-1}(-2)$ (d) none of these

SECTION - 3

This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D). Column II has five entries (P), (Q), (R), (S) and (T). Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 5 matrix whose layout will be similar to the one as shown.

For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D).

(A)	(P)	(Q)	(R)	(S)	(T)
(B)	(P)	(Q)	(R)	(S)	(T)
(C)	(P)	(Q)	(R)	(S)	(T)
(D)	(P)	(Q)	(R)	(S)	(T)

Marking scheme : For each entry in Column I, +2 If only the bubble(s) corresponding to all the correct match(es) is (are) darkened; 0 If none of the bubbles is darkened; –1 In all other cases.

19. Match the following :

	Column I		Column II
A.	Maximum value of $5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)-1$ is	P.	1
B.	$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right)=t$, then $\tan t$ =	Q.	6
C.	If $3\cos\theta - 4\sin\theta = 5$, then $3\sin\theta + 4\cos\theta$ =	R.	2
D.	If $\sin A \sin B \sin C + \cos A \cos B$ = 1, then $2\cos(A - B)$ =	S.	0

20. Match the following:

	Column I		Column II
A.	In a ΔABC, $\cos A + \cos B + \cos C$ is equal to	P.	$\frac{a^2+b^2+c^2}{\Delta^2}$
B.	$\frac{r_1-r}{a}+\frac{r_2-r}{b}+\frac{r_3-r}{c}=$	Q.	$1+\frac{r}{R}$
C.	$\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=$	R.	$\frac{r_1+r_2+r_3}{s}$
D.	$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}$	S.	$\frac{1}{r}-\frac{1}{2R}$

PAPER-II

SECTION-1

This section contains eight questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. If l is the shortest distance between the lines $\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$ and $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$, then determine $[l]$ ($[x]$ denotes G.I.F)

2. Let $\vec{a}$ and $\vec{b}$ be unit vectors. If $\vec{c}$ is a vector such that $\vec{c}+(\vec{c}\times\vec{a})=\vec{b}$, then the maximum value of $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ is $A\times 10^{-1}$. Find A.

3. The straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular then the value of $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}$ is

4. If the distance of the point $B(\hat{i}+2\hat{j}+3\hat{k})$ from the line which is passing through $A(4\hat{i}+2\hat{j}+2\hat{k})$ and which is parallel to the vector $\vec{c}=2\hat{i}+3\hat{j}+6\hat{k}$ is λ then value of $\lambda^2 - 1$ is

5. Find the number of real solutions of the equation $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$.

6. Number of roots of the equation $\cos(\sin x)=\frac{1}{\sqrt{2}}$, $(0 < x < \pi)$ is

7. Let x, y, z be real numbers such that $\cos x + \cos y + \cos z = 0$ and $\cos 3x + \cos 3y + \cos 3z = 0$ then find the maximum value of $\cos 2x \cos 2y \cos 2z$.

8. If $x + y = 2\pi/3$ and $\sin x/\sin y = 2$, then the number of values of $x \in [0, 4\pi]$ is

SECTION-2

This section contains eight questions. Each question has four options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct options in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 if none of the bubbles is darkened, –2 In all other cases.

9. Y-coordinate of the point of intersection of the curves is, $y = \cos x$, $y = \sin 3x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ is

(a) $\cos\left(\frac{\pi}{8}\right)$ (b) $\frac{1}{\sqrt{2}}$
(c) $\sin\left(\frac{\pi}{8}\right)$ (d) $-\frac{1}{\sqrt{2}}$

10. The satisfactory value of x, where $x \in (-\pi, \pi)$ for the given equation,
$(\sqrt{3}\sin x+\cos x)^{\sqrt{(\sqrt{3}\sin 2x-\cos 2x+2)}}=4$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{5\pi}{3}$ (c) $-\frac{2\pi}{5}$ (d) $-\frac{\pi}{3}$

11. If $\left(\cos^2 x+\frac{1}{\cos^2 x}\right)(1+\tan^2 2y)(3+\sin 3z)=4$, then

(a) x may be a multiple of π
(b) x cannot be an odd multiple of π
(c) y can be a multiple of $\pi/2$
(d) z can be a multiple of $\pi/2$

12. The values of 'x' which satisfy the equation $2(\cos x + \cos 2x) + \sin 2x(1 + 2\cos x) = 2\sin x$, and $-\pi \le x \le \pi$ are

(a) $\pi, -\pi$ (b) $-(\pi/3), (\pi/3)$
(c) $-\pi/2$ (d) none of these

13. For $\theta \in [0, 2\pi]$, the values of 'θ' and 'y' satisfying the inequality, $2^{\frac{1}{\sin^2\theta}}\cdot\sqrt{y^2-2y+2}\le 2$ are

(a) $\left(\frac{\pi}{4}, -1\right)$ (b) $\left(\frac{\pi}{2}, 1\right)$
(c) $\left(\frac{3\pi}{2}, 1\right)$ (d) $\left(\frac{\pi}{4}, 1\right)$

14. The positive integral solutions (x, y) of $\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ are

(a) (1, 2) (b) (2, 1)
(c) (2, 7) (d) (2, 2)

MPP-5 CLASS XII ANSWER KEY

1.	(b)	2.	(b)	3.	(d)	4.	(d)	5.	(c)
6.	(b)	7.	(a,b,c)	8.	(c,d)	9.	(b,c)	10.	(b,c)
11.	(a,b,c,d)	12.	(a,b,c,d)	13.	(b,c)	14.	(a)	15.	(d)
16.	(c)	17.	(3)	18.	(0)	19.	(0)	20.	(2)

15. Solving the equation $\frac{\sqrt{3}}{2}\sin x - \cos x = \cos^2 x$, we get

(a) $x = (2n + 1)\pi$ (b) $x = 2n\pi + \frac{\pi}{3}$

(c) $x = 2n\pi - \frac{\pi}{3}$ (d) none of these

16. The roots of the equation $\left(\sin\frac{\pi}{6}\right)^{\cot 3x + 2\cot x} =$ $\sin^2(2\pi - x) - \cos(\pi - x)\times\sin\left(\frac{\pi}{2}+x\right)$ are

(a) $x = n\pi + \tan^{-1}\sqrt{\frac{7}{5}}$

(b) $x = n\pi - \tan^{-1}\sqrt{\frac{7}{5}}$

(c) $x = \tan^{-1}\sqrt{\frac{5}{7}}$

(d) none of these

SECTION-3

This section contains two paragraphs. Based on each paragraph, there will be two questions. Each question has four options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened ; 0 If none of the bubbles is darkened ; – 2 In all other cases.

Paragraph for Q. No. 17 & 18

Let 'k' be the length of any edge of a regular tetrahedron. (A tetrahedron whose edges are all equal in length is called a regular tetrahedron). The angle between a line and a plane is equal to the complement of the angle between the line and the normal to the plane whereas the angle between two planes is equal to the angle between the normals. Let 'D' be the origin be one of the vertex and A, B and C vertices with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively of the regular tetrahedron.

17. The angle between any edge and a face not containing the edge is

(a) $\cos^{-1}(1/2)$ (b) $\cos^{-1}(1/4)$

(c) $\cos^{-1}(1/\sqrt{3})$ (d) $\pi/3$

18. The angle between any two faces is

(a) $\cos^{-1}(1/\sqrt{3})$ (b) $\cos^{-1}(1/4)$

(c) $\pi/3$ (d) none of these

Paragraph for Q. No. 19 & 20

The circle touching BC and the two sides AB and AC produced of a triangle ABC is called the described circle opposite angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the described circles opposite angles B and C, respectively. They are known as the ex-radii of triangle ABC. Here Δ is the area of a triangle, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$. Also R is the radius of the circumcircle of a triangle ABC and r is the radius of the circle inscribed in triangle ABC.

19. Suppose in a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is

(a) $\frac{1}{3}$ (b) $\frac{16}{7}$ (c) $\frac{3}{14}$ (d) $\frac{8}{19}$

20. Suppose in a triangle ABC, $8R^2 = a^2 + b^2 + c^2$, then the triangle ABC is

(a) right angled (b) isosceles

(c) equilateral (d) none of these

ANSWER KEY

PAPER-I

1. (2) 2. (2) 3. (1) 4. (2) 5. (8)
6. (5) 7. (6) 8. (4) 9. (a, b) 10. (b, d)
11. (a, b, c, d) 12. (c) 13. (a, b, c)
14. (b, c) 15. (b, c) 16. (a, c, d)
17. (b, c, d) 18. (a, b)
19. (A) → (Q), (B) → (P), (C) → (S), (D) → (R)
20. (A) → (Q), (B) → (R), (C) → (S), (D) → (P)

PAPER-II

1. (6) 2. (5) 3. (0) 4. (9) 5. (1)
6. (2) 7. (0) 8. (4) 9. (a, b, c)
10. (a, b) 11. (a, c) 12. (a, b, c)
13. (b, c) 14. (a, c) 15. (a, b) 16. (a, b) 17. (c)
18. (d) 19. (b) 20. (a)

Solution Sender of Maths Musing

SET-165

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SET-166

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MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $f(x)=\begin{cases}|x|, & \text{when } x\le 2\\ [x], & \text{when } x>2\end{cases}$, then

(a) $\lim\limits_{x\to 2^-} f(x)=-2$ (b) $\lim\limits_{x\to 2^+} f(x)=-2$

(c) $\lim\limits_{x\to 2^+} f(x)=f(2)$

(d) $\lim\limits_{x\to 2} f(x)$ does not exist

2. If $n\in N$, then $\lim\limits_{x\to\infty}\frac{x^n}{e^x}=0$

(a) when n is even only (b) for no value of n

(c) for all values of n (d) when n is odd only

3. If $y=\sin^{-1}\left[\frac{1-x^2}{1+x^2}\right]$, then $\frac{dy}{dx}=$

(a) $\frac{-2}{1+x^2}$ (b) $\frac{2}{1+x^2}$ (c) $\frac{1}{2+x^2}$ (d) $\frac{2}{2-x^2}$

4. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x=0$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

5. If $y=\tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$ then $\frac{d^2y}{dx^2}$ is

(a) 2 (b) 1 (c) 0 (d) -1

6. Let $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ;x\neq 0\\ 0 & ;x=0\end{cases}$

Test whether $f(x)$ is differentiable at $x=0$. Is it continuous at $x=0$?Justify

7. A triangle ABC, right angle at C, with $CA=b$ and $CB=a$, moves such that the angular points A and B slide along x-axis and y-axis respectively. Find the locus of C.

8. Prove that $\sin\theta\cdot\sec 3\theta=\frac{1}{2}(\tan 3\theta-\tan\theta)$ and hence find the sum to 'n' terms of the series $\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\theta+\sin 3^2\theta\cdot\sec 3^3\theta+\ldots$

9. Consider a real valued function $f(x)$ satisfying $2f(xy)=(f(x))^y+(f(y))^x\ \forall\ x,y\in R$ and $f(1)=p$ where $p\neq 1$ then find $(p-1)\sum\limits_{r=1}^{n}f(r)$.

10. A line makes angles $\alpha,\beta,\gamma,\delta$ with four diagonals of a cube. Prove that $\sum\limits_{r\in\{\alpha,\beta,\gamma,\delta\}}\cos^2(r)=\frac{4}{3}$.

SOLUTIONS

1. (c) : $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{h\to 0}[2+h]=2=f(2)$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{h\to 0}|2-h|=2$

Hence limit exists

2. (c) : Use L′ Hospital rule, $\lim\limits_{x\to\infty}\frac{x^n}{e^x}=\frac{n!}{e^\infty}=0$

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3. **(a) :** Put $x = \tan\theta \Rightarrow y = \frac{\pi}{2} - 2\theta$

4. **(b) :** $p = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ Put $x = \tan\theta$

$\therefore\ p = \frac{\theta}{2} \Rightarrow \frac{dp}{dx} = \frac{1}{2(1+x^2)}$ ...(i)

For $q = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ Put $x = \sin\phi$

$\therefore\ \frac{dq}{dx} = \frac{2}{\sqrt{1-x^2}}$ ...(ii)

From (i) and (ii)

$$\frac{dp}{dq} = \frac{\frac{1}{2\left(1+x^2\right)}}{\frac{2}{\sqrt{1-x^2}}} \quad \therefore \left.\frac{dp}{dq}\right|_{x=0} = \frac{1}{4}$$

5. **(c) :** In given function put $\log x^2 = \tan\theta$

Therefore, $y = \frac{\pi}{4} + \tan^{-1}3$. Hence, $\frac{d^2y}{dx^2} = 0$

6. $f(x) = \begin{cases} xe^{\frac{-2}{x}} & , x > 0 \\ 0 & , x = 0 \\ x & , x < 0 \end{cases}$

$\therefore\ f(0) = 0$

L.H.D= $f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-h}{-h} = 1$

R.H.D= $f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{he^{\frac{-2}{h}}}{h} = e^{-\infty} = 0$

$\because$ L. H. D $\neq$ R. H. D

$\therefore$ f is not differentiable at $x = 0$

$\because$ L. H. D and R. H. D are exist

$\therefore$ f is continuous at $x = 0$

7. Let $A = (p, 0), B = (0, q)$

Let $C(h, k)$ be any point on the locus

$CB = a = \sqrt{h^2 + (k-q)^2}$ ...(i)

$CA = b = \sqrt{(h-p)^2 + k^2}$ ...(ii)

$AB = \sqrt{p^2 + q^2}$ ...(iii)

$\because\ \angle C = 90°$

$\therefore\ AB^2 = AC^2 + BC^2 \Rightarrow p^2 + q^2 = a^2 + b^2$ ...(iv)

From (i) and (ii)

$q = k \pm \sqrt{a^2 - h^2}$, $p = h \pm \sqrt{b^2 - k^2}$

$\therefore$ From (iv), locus is $bx \pm ay = 0$

8. $\frac{\sin\theta}{\cos 3\theta} = \frac{\cos\theta \cdot \sin\theta}{\cos\theta \cdot \cos 3\theta}$

$= \frac{1}{2}\frac{\sin(3\theta - \theta)}{\cos\theta \cdot \cos 3\theta} = \frac{1}{2}(\tan 3\theta - \tan\theta)$

$\therefore\ \sum_{r=1}^{n} \sin 3^{r-1}\theta \cdot \sec 3^r\theta$

$= \frac{1}{2}\sum_{r=1}^{n}\left(\tan 3^r\theta - \tan 3^{r-1}\theta\right) = \frac{1}{2}\left[\tan 3^n\theta - \tan\theta\right]$

9. $2f(xy) = (f(x))^y + (f(y))^x\ \forall\ x, y \in R$

Put $y = 1$

$\Rightarrow 2f(x) = f(x) + (f(1))^x \Rightarrow f(x) = p^x$

$\therefore\ \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} p^r = \frac{p^{n+1} - p}{p-1}$

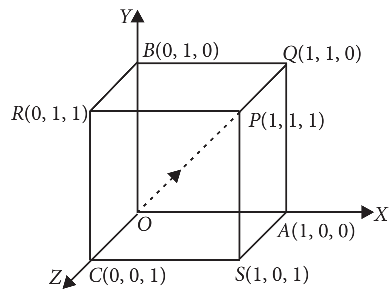
$\Rightarrow (p-1)\sum_{r=1}^{n} f(r) = \left(p^{n+1} - p\right)$

10. $d.c\text{'s of } \overrightarrow{OP} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$d.c\text{'s of } \overrightarrow{AR} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$d.c\text{'s of } \overrightarrow{BS} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$d.c\text{'s of } \overrightarrow{CQ} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$



Let l, m, n be the $d.c$'s of required line

$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= \frac{1}{3}\left((l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2 + (-l+m+n)^2\right)$

$= \frac{4}{3}$

MATHS MUSING

SOLUTION SET-166

1. (c): Let $f(x, y) = a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3$.

$f(\pm 1, 0) = 0 \Rightarrow a_3 = 0, a_6 = -a_1$

$f(0, \pm 1) = 0 \Rightarrow a_5 = 0, a_9 = -a_2$

Also, $a_8 = 0, a_7 = -a_4$

$f(2, 2) = 0 \Rightarrow a_4 = -\frac{3}{2}(a_1 + a_2)$

$$\text{Now } f(x, y) = \frac{a}{2x}(1-x)(2+2x-3y) + \frac{a_2}{2}y(2-3x+3x^2-2y^2)$$

$f(x, y) = 0$ if $y = \frac{2}{3}(1+x)$ and

$$2 - 3x + 3x^2 - \frac{8}{9}(1+x)^2 = 0 \Rightarrow a + b = \frac{21}{19}$$

2. (b): Let $C = \frac{\pi}{2}$ in triangle ABC.

$\Delta = rs \Rightarrow rc = ab - (a+b)r$.

On squaring, we get

$(a - 2r)(b - 2r) = 2r^2 = 2{\cdot}2011^2$.

Number of triangles is one half of the number of divisors of $2^1 \cdot 2011^2$, which is $\frac{1}{2}(1+1)(2+1) = 3$.

3. (c): $C = \left(\frac{1}{2}, 0\right)$, $P = (\sqrt{2}\cos\theta, \sin\theta)$

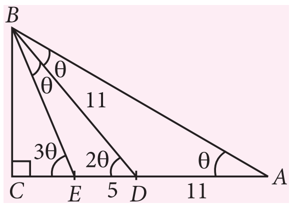
$$CP^2 = \left(\sqrt{2}\cos\theta - \frac{1}{2}\right)^2 + \sin^2\theta = \frac{5}{4} - \sqrt{2}\cos\theta + \cos^2\theta$$

Derivative $= \sqrt{2}\sin\theta - 2\sin\theta\cos\theta = 0$

$\Rightarrow$ CP is min. for $\cos\theta = \frac{1}{\sqrt{2}}$ $\therefore$ Largest radius is $\frac{\sqrt{3}}{2}$.

4. (c): In triangle EDB, $\frac{11}{\sin 3\theta} = \frac{5}{\sin\theta}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}$



In triangle BCD,

$BC = BD\sin 2\theta$

$$= 11 \cdot 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = \frac{44}{5}$$

5. (c): Differentiating the given curve, we get

$2y' + 25x^4 - 30x^2 + 1 = 0, y'(0) = -\frac{1}{2}$

The normal at $(0, -3)$ is $y = 2x - 3$.

Eliminating y with the given equation, we get

$x(x^2 - 1)^2 = 0 \Rightarrow x = 0, 1, -1$

6. (a,c,d): $15N = \sum_{r=2}^{7}\binom{15}{r} = \frac{1}{2}\sum_{r=2}^{13}\binom{15}{r}$

$\therefore$ $30N = 2^{15} - 2(1 + 15) = 2^{15} - 32$

$\Rightarrow$ $5N = 5456 = 2^4 \cdot 11 \cdot 31$

7. (d) **8. (a)**

9. (3): The differential equation satisfied by $(x - c)^2 + y^2 = 1$ is $y^2(y')^2 = 1 - y^2$

Replacing y' by $-\frac{1}{y'}$ we get

$$\frac{(1-y^2)(-y')^2}{y^2} = 1 \Rightarrow \frac{\sqrt{1-y^2}}{y}dy = -dx$$

Integrating we get

$$x + c = -\sqrt{1-y^2} + \ln\left(\frac{1+\sqrt{1-y^2}}{y}\right)$$

$x = 0, y = 1 \Rightarrow c = 0$

$$\therefore\ y = \frac{\sqrt{3}}{2} \Rightarrow x = -\frac{1}{2} + \ln\sqrt{3} \Rightarrow e^{2x+1} = 3$$

10. (b) : (P) → 4; (Q) → 2; (R) → 3; (S) → 3

$$\text{(P)}\ \int_{1/e}^{\tan x}\frac{tdt}{1+t^2} + \int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}, \left(t = \frac{1}{u}\right)$$

$$= \int_{1/e}^{\tan x}\frac{tdt}{1+t^2} + \int_{\tan x}^{e}\frac{udu}{(1+u^2)} = \int_{1/e}^{e}\frac{tdt}{1+t^2} = 1.$$

$$\text{(Q)}\ \int_0^{\pi/2}\frac{dx}{(\sqrt{\sin x} + \sqrt{\cos x})^4}, (\tan x = t)$$

$$= \int_0^{\infty}\frac{dt}{(\sqrt{t}+1)^4} = 2\int_1^{\infty}\left(\frac{1}{u^3} - \frac{1}{u^4}\right)du = \frac{1}{3}.$$

$$\text{(R)}\ I_1 = \int_0^1\frac{\tan^{-1}x}{x}dx = \int_0^{\pi/4}\frac{2\theta}{\sin 2\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{x}{\sin x}dx = \frac{1}{2}I_2 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$$

$$\text{(S)}\ F(e) = \int_1^e\frac{\ln t}{1+t}dt + \int_1^{1/e}\frac{\ln t}{1+t}dt, \left(t = \frac{1}{u}\right)$$

$$= \int_1^e\frac{\ln t}{1+t}dt + \int_1^e\frac{\ln u}{u(1+u)}du = \int_1^e\frac{\ln t}{(t+1)}\left(1+\frac{1}{t}\right)dt = \frac{1}{2}.$$

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YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

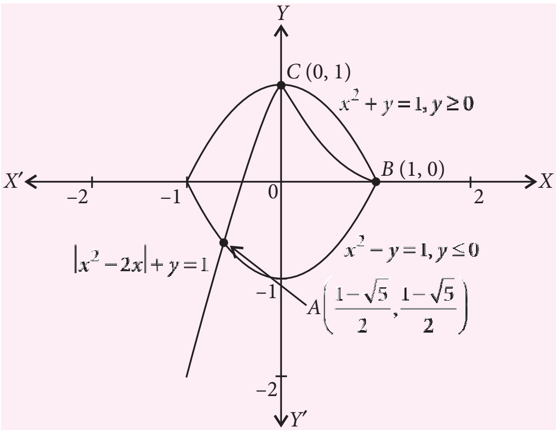
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. Find the number of real solutions of the equations $|x^2 - 2x| + y = 1,\ x^2 + |y| = 1$?

Abhishek, Kolkata

Ans. We will consider the four cases separately:



(a) $x^2 - 2x \geq 0, y \geq 0$: Here, we have to solve $x^2 - 2x + y = 1, x^2 + y = 1$

The unique solution is $x = 0, y = 1$.

(b) $x^2 - 2x \leq 0, y \geq 0$: Here, we have to solve $-x^2 + 2x + y = 1, x^2 + y = 1$

We have $-2x^2 + 2x = 0$ and $(x, y) = (0, 1)$ or $(1, 0)$.

The new solution is $(1, 0)$.

(c) $x^2 - 2x \geq 0, y \leq 0$: Here, we have to solve $x^2 - 2x + y = 1, x^2 - y = 1$

Here $x^2 - x - 1 = 0, x = y$. Solving the quadratic in x we get $x = \frac{1 \pm \sqrt{5}}{2}$. $y \leq 0$,

$$\Rightarrow\ (x, y) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

(d) $x^2 - 2x \leq 0, y \leq 0$: Here, we have to solve $-x^2 + 2x + y = 1, x^2 - y = 1$

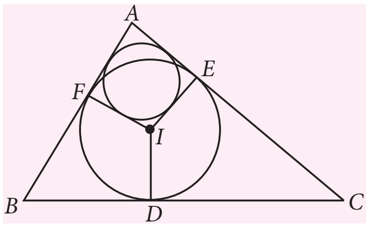
Here again the unique solution is $(1, 0)$. Thus there are three solutions.

2. Let ABC be a triangle with incentre I and inradius r. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2, r_3 are the radii of the circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respectively, then prove that

$$\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$

Vishal, U.P.

Ans.



The exaggerated view of the quadrilateral $AFIE$ and its incircle is shown below where $IF = r$ and $MI_1 = NI_1 = r_1$.

From the figure, we have

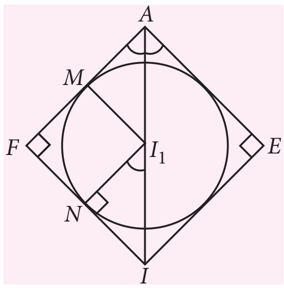
$\angle MAI_1 = \angle NI_1 I = A/2$

and $IN = IF - NF = IF - MI_1 = r - r_1$

Therefore, we have

$$\cot\left(\frac{A}{2}\right) = \frac{NI_1}{NI} = \frac{r_1}{r-r_1}$$

Similarly, we have



$$\cot\left(\frac{B}{2}\right) = \frac{r_2}{r-r_2}$$

and $\cot\left(\frac{C}{2}\right) = \frac{r_3}{r-r_3}$

Now, we have

$$\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$

i.e. $\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)$

$$= \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

i.e. $$\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}$$

which is the desired result.

◈◈